SHORT STUDY SERIES
Nº 40, dezembro de 2024

FINANÇAS

FGV INVEST

DIFFUSION PROCESSES 1

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CENTRO DE TREINAMENTO EM PROCESSOS DE INVESTIMENTO

Diffusion Processes 1

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ABSTRACT

This text explores ideas related to diffusion processes, specifically highlighting the roles of the scale function and speed density. We focus on an Ornstein–Uhlenbeck model as a framework for answering practical questions about the behavior of a stochastic process—such as a trading strategy—including when it may exit a predefined region and where optimal entry and exit points might lie. To illustrate these concepts, we present a simplified pairs trading example, demonstrating how the theoretical components translate into actionable insights.

Key-words: Diffusion processes, Ornstein-Uhlenbeck, Scale function, Speed density.

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1. INTRODUCTION

Diffusion models represent a class of stochastic processes that are highly flexible and may be particularly well-suited for specific applications in financial data analysis. For these models, the optimal martingale estimating functions can be derived explicitly. For detailed discussions, refer to "The Pearson diffusions: A class of statistically tractable diffusion processes" (Sørensen, Forman and Margit, 2007) and earlier works by Sørensen and works by Salminen, for example “One-Dimensional Diffusions and Their Exit Spaces” (P. Salminen, 1984).

2. THE PROCESS

Consider a diffusion process as a stationary solution to the following Stochastic Differential Equation (SDE):

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

Where drift (μ) and volatility (σ) functions are 'well behaved' and continuous on a subset of real numbers.

A notable transformation arises from the **scale function** and **speed density**, which, broadly speaking, enable the construction of local martingales:

$$s(x) = \int_0^x \left[e^{-2\int_0^y \frac{\mu}{\sigma^2}(z)dz} \right] dy$$

$$m(dx) = \frac{2}{\sigma^2(x)} \left[e^{2\int_0^x \frac{\mu}{\sigma^2}(z)dz} \right] dx$$

Local martingales are noteworthy because for any discrete-time ***P***-local martingale **S**, there exists a probability measure ***Q*** equivalent to ***P*** (denoted by ***Q***~***P***) under which **S** is a true martingale and then it implies that at any given time, the conditional expected value of the future outcome (given all prior information) equals its current value.

3. THE SOLUTION

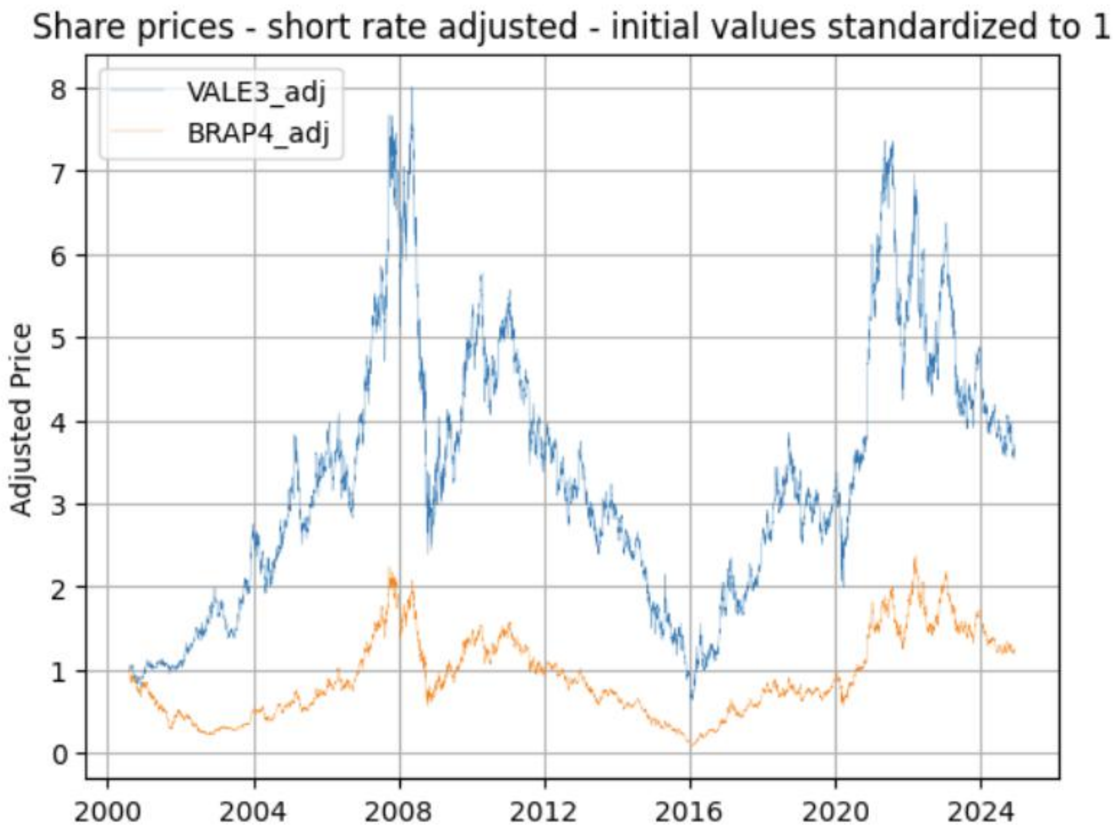
Assuming that the total cost involved into one trading cycle is *c* then, within a previously defined domain (l,r); in a **significant** number of trading cycles, an optimal solution, **y***, to the function **expected return** can be determined by an equation of the following type:

$$y^* = \arg\max_{y \in (l,r)}(\mathbb{E}[return]);\ y^* \geq 0;\ y^* \in \left(\frac{c}{2}, r\right)$$

In this particular case, the maximization will include both *s(x)* and *s'(x),* above explicited.

4. WHY DOES IT MATTER?

A fair question to ask is: How does the initial SDE relate to real-world financial data? Let's clarify this with an example. Consider the following chart, which uses adjusted closing prices of two financial assets (share prices of BRAP4.SA and VALE.SA). The short-term interest rate has also been deducted from the prices to account for time value effects.



Using the **Ornstein-Uhlenbeck (O-U)** process as an illustrative example, a special case of the general SDE, the drift and volatility functions are specified as:

$$O - U \ \Rightarrow \ \mu(X_t) = \mu(\theta - X_t); \ \sigma(X_t) = \sigma$$

Applying simplified estimators, such as Ordinary Least Squares (OLS), the parameters μ, θ, and σ are found to fall within the following intervals: [8,12]; [-0.02,0.01]; [0.25,0.65]. These estimators vary over time due to market conditions and sample data.

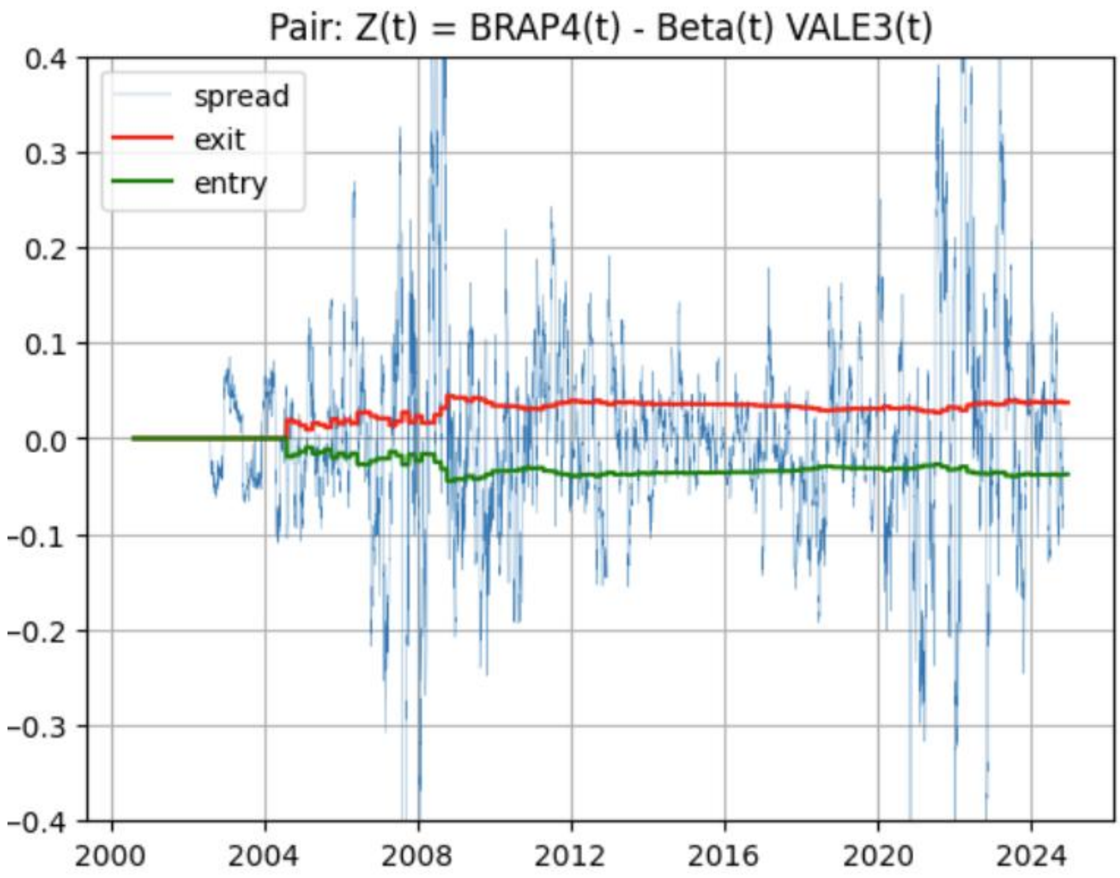
5. ENTRY AND EXIT REGIONS

Using the derived solution, the **entry** and **exit** regions for the given parameter intervals are calculated as:

- Entry: [-0.05,-0.01]

- Exit: [0.01,0.05]

Incorporating these regions into a spread chart:



Contrary to common intuition, the optimal **entry (buy)** and **exit (sell)** regions are **not** predetermined by a fixed number of standard deviations. Instead, they are determined dynamically based on the function maximization solution derived from the diffusion model.

6. CONCLUSION

This framework demonstrates how diffusion models can offer a mathematically rigorous foundation for defining trading strategies and informing decision-making in financial markets. By applying concepts such as martingales, speed densities, and scale functions, practitioners can extract valuable, data-driven insights adapted to evolving market conditions.

Please note that the stocks mentioned in the example were selected purely for illustrative purposes and may not have constituted a profitable pairing throughout the entire period examined.

It's also important to remember that any model is merely a simplified representation of reality—a stylized lens through which we attempt to capture the most significant aspects of a phenomenon. In addition, raw data often requires preprocessing before it can be reliably integrated into a modeling framework.

Enjoy exploring these models and discovering their potential applications!

REFERENCES

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