Trading Dynamics in Decentralized Markets with Adverse Selection

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Abstract

We study a dynamic, decentralized lemons market with one–time entry and characterize its set of equilibria. Our framework offers a theory of how markets suffering from adverse selection recover over time endogenously; given an initial fraction of lemons, the model provides sharp predictions about the evolution of prices and the composition of assets in the market. Comparing economies in which the initial fraction of lemons varies, we study the relationship between the severity of the lemons problem and market liquidity. We use this framework as a laboratory to analyze the Public–Private Investment Program for Legacy Assets, a policy implemented during the recent financial crisis in order to restore market liquidity. We find that, depending on the fraction of lemons in the market, such a program can speed up or slow down market recovery. More generally, our framework highlights that the success of an intervention in a lemons market depends on both its size and duration.

Keywords: Adverse Selection, Decentralized Trade, Liquidity, Market Freeze and Recovery

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1 Introduction

Since the seminal work of Akerlof (1970), it is well known that the introduction of low quality assets, or “lemons,” into a market with asymmetrically informed buyers and sellers can disrupt trade; the typical result is that sellers with high quality assets are unwilling to sell at depressed prices, and thus only low quality assets are exchanged in equilibrium. Given this result, the problem of adverse selection is often used to explain why the market for high quality assets can break down or freeze. However, perhaps surprisingly, much less is known about how and when the exchange of these assets resumes, or how this market thaws.

In this paper, we develop a simple model of trade under adverse selection and use it to study how the severity of the lemons problem (i.e., the initial fraction of lemons in the market) affects the patterns of trade over time. In contrast to much of the existing literature, in which unfreezing a market requires an exogenous event or intervention, we incorporate several natural features of actual asset markets that allow this process of recovery to occur endogenously. Thus, given any initial fraction of lemons, our model delivers sharp predictions about the length of time it takes for the market to recover, and how prices and the composition of assets remaining in the market behave over this horizon.

We find that the patterns of trade depend systematically on the initial fraction of lemons. In particular, when the lemons problem is mild (i.e., this fraction is small), trades are executed quickly and at relatively uniform prices. However, when the lemons problem is more severe, trade can take a substantial amount of time and the terms of trade can vary significantly, both across agents and over time. We also characterize how the severity of the lemons problem affects the expected amount of time it takes to sell a high quality asset, which we interpret as a measure of the market’s illiquidity; a liquid market is one where sellers can quickly find a buyer to purchase their high quality asset (at an acceptable price), whereas an illiquid market is one where this process takes a long time. In this sense, the theory presented here provides a novel theory of liquidity based on adverse selection.

Given that our framework describes explicitly how markets can recover over time on their own, it also provides a natural setting to analyze how policies aimed at restoring liquidity
can speed up (or slow down) this process. We provide a specific example related to the recent financial crisis and illustrate how our environment can provide unique—and perhaps counter–intuitive—insights into the efficacy of such policy interventions.

We take as a starting point the classic lemons market of Akerlof (1970) and make a few simple modifications. First, in order to study how a frozen market can recover over time, the environment must be dynamic and equilibria must be non–stationary. Therefore, we consider a discrete–time, infinite–horizon model in which a fixed set of buyers and sellers have the opportunity to trade in each period. In addition, we assume that agents permanently exit the market after trading, and there are no new entrants. As a result, a central aspect of our analysis is how the composition of assets remaining in the market evolves over time, and how this interacts with agents’ incentives to trade at a particular point in time. Thus, in our model there is a formal sense in which trade may be sluggish because agents are waiting for market conditions to improve, which seems to be an important feature of many frozen markets that cannot be captured in a static or stationary setting.

Second, we focus our analysis on markets in which trade is decentralized; in contrast to the competitive paradigm, where agents are bound by the law of one price, we assume that buyers and sellers are matched in pairs, and that they decide bilaterally whether to trade and at what price. This assumption is consistent with the trading structure in many important asset markets, such as the markets for asset–backed securities, corporate bonds, derivatives, real estate, and even certain equities.\footnote{By now, the literature on decentralized or “over–the–counter” asset markets has grown quite large; see, e.g., Duffie et al. (2005), Vayanos and Weil (2008), and Lagos and Rocheteau (2009).}

There are two reasons why these modifications allow for the eventual exchange of high quality assets. First, there are two mechanisms that can adjust to facilitate trade: the price and, equally important, the time at which a transaction takes place. Second, agents with different quality assets are allowed to trade at different prices.\footnote{See Blouin (2003) and Moreno and Wooders (2010) for more extensive comparisons between centralized and decentralized exchange in a dynamic setting with adverse selection.} In the context of this environment, we then ask the following questions. Are all assets—and in particular high quality assets—eventually bought and sold? If so, how long does it take? How does the
presence of low quality assets affect the expected amount of time it takes to sell high quality assets? How do prices and the composition of assets in the market evolve over time?

Before we report our findings, it is helpful to describe the model in more detail. The economy starts at $t = 0$ with an equal measure of buyers and sellers. A fraction $q_0 \in (0, 1)$ of sellers possess a single high quality asset, and the remainder possess a single low quality asset. The quality of a seller’s asset is private information. In each period $t = 0, 1, 2, \ldots$, all agents receive a stochastic discount factor shock, and then buyers and sellers in the market are randomly and anonymously matched in pairs. Once matched, buyers make one of two exogenously set price offers: a high price (that in equilibrium is accepted by all sellers) or a low price (that in equilibrium is only accepted by impatient sellers with the low quality asset). If a seller accepts the buyer’s offer, trade ensues and the pair exits the market; if the seller rejects, the agents remain in the market. There are gains from trade in every match, so that the efficient outcome is for all trade to take place immediately.

Within this environment, we completely characterize the equilibrium set for all $q_0 \in (0, 1)$, and use this characterization to study the effects of asymmetric information on the patterns of trade. First, given any $q_0$, we show that all assets are bought and sold—the market clears—in a finite number of periods. The patterns of trade are such that average price offers and the average quality of assets in the market increase over time until, eventually, the average quality is high enough that all remaining buyers offer the high price, and the market clears. However, the amount of time it takes until the market clears depends crucially on the initial fraction of high quality assets: the equilibrium characterization involves partitioning the interval $(0, 1)$ based on how many periods of trade, $k$, it takes before all assets are bought and sold, for a given $q_0 \in (0, 1)$. Figure 1 below depicts a typical (very simple) partition.

We highlight two interesting features of this equilibrium characterization. First, there is a natural monotonicity to the equilibrium set: as $q_0$ gets smaller, it takes longer for the market to clear. We also derive the expected amount of time it takes to sell a high quality

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3We discuss these assumptions in more detail below, when we introduce the environment. Moreover, we provide an extensive discussion in Section 7 about the advantages of restricting buyers to offer one of two prices, and the robustness of our results to alternative specifications.
Figure 1: Number of Periods ($k$) Before Markets Clear for $q_0 \in (0, 1)$

asset, which measures the extent to which the market for these assets is illiquid, and analyze the relationship between this measure of illiquidity and the initial fraction of lemons. It is in this sense that our model provides a theory of endogenous liquidity that varies systematically across states of the world and over time.

Second, note that the equilibrium regions in Figure 1 overlap: for some values of $q_0$, there are multiple equilibria that take different amounts of time for the market to clear. This multiplicity is driven by a complementarity between buyers’ actions. When other buyers offer the high price, average quality in the ensuing period does not change, since sellers with both high and low quality assets accept the high price in equal proportion. This gives buyers less incentive to wait for future periods to trade and more incentive to offer a high price now. On the other hand, when other buyers are offering the low price, a larger proportion of sellers with low quality assets accept this offer relative to sellers with high quality assets, and average quality in the future increases. This provides buyers less incentive to offer a high price and trade immediately. The existence of multiple equilibria for a given $q_0$ suggests that coordination failures can also contribute to illiquidity in dynamic, decentralized market settings with adverse selection.

As pointed out above, since our model provides an explicit theory of how markets recover on their own, it also provides a natural framework to analyze policies aimed at speeding up this process. As a leading example, we consider a stylized version of a policy implemented in the market for asset–backed securities in the wake of the financial crisis that began in 2007, the so-called Public–Private Investment Program for Legacy Assets. This policy provided

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4Our model captures many of the essential features of this market: trade is decentralized, the fall of housing prices implied substantial heterogeneity in the value of these assets, and in many cases sellers had more information about these assets than potential buyers. We argue each of these points in greater detail
non–recourse loans to buyers willing to purchase these securities, thus reducing the buyers’ downside exposure should they discover that they acquired a lemon.

In the context of either static or stationary models of adverse selection, a reduction of downside risk would almost surely ease the lemons problem and help restore liquidity, as this provides buyers with the incentive to offer higher prices. Within the context of our model, we show such a policy has ambiguous effects on market recovery. Intuitively, an increase in the incentive of buyers to offer higher prices increases both current and future payoffs for sellers holding low quality assets. If the increase in future payoffs is greater than the increase in current payoffs, then the owners of low quality assets have an incentive to delay trade, which slows the market’s recovery, making high quality assets less liquid and reducing welfare. Thus, our model highlights that the timing—and in particular the duration—of any intervention can be just as important as its size. This is a point that has been largely ignored by the literature, in part because most existing work abstracts from non–stationary dynamics (or restricts attention to one–time interventions).

The rest of the paper is organized as follows. After discussing the related literature below, we introduce the environment in Section 2. In Section 3, we establish some basic properties of equilibria, and in Section 4, we provide a complete characterization of the equilibrium set. In Section 5, we discuss three aspects of our equilibrium characterization: the relationship between liquidity and the lemons problem, the dynamics of trade, and the multiplicity of equilibria. In Section 6, we discuss our application to the market for asset–backed securities. In Section 7, we discuss some of our assumptions, including the restriction that buyers can only offer one of to two prices. Section 8 concludes.

As it turns out, many of our results are true in a more general setting in which buyers are free to offer any price they wish. The assumption of two prices, however, allows for a simple, complete analytical characterization of the equilibrium set. Given the complexity, in general, of non–stationary environments, we think that the gains from a having a simple in Section 6.

\footnote{Indeed, there is a tradition of exploiting the tractability of a two–price framework within the literature on matching and bargaining in the presence of asymmetric information, both in stationary and non–stationary environments; see, e.g., Wolinsky (1990), Samuelson (1992), and Blouin and Serrano (2001).}
equilibrium characterization outweigh the losses: the results here highlight the key mechanisms at work, and provide a parsimonious environment to think about the inefficiencies wrought by asymmetric information, as well as the effects of interventions. For the sake of completeness, however, we also derive and present many of the results in the more general setting in the Supplementary Appendix.

Related Literature

Our work builds on the literature that studies dynamic, decentralized markets with asymmetric information and interdependent values. The majority of this literature restricts attention to stationary equilibria; see, for example, Inderst (2005), Moreno and Wooders (2010), and the references therein. A notable exception is Blouin (2003), who analyzes non-stationary equilibria. In all of these papers, the primary focus is to determine what happens to equilibria in a decentralized trading environment as market frictions vanish. In contrast, we completely characterize the set of (non-stationary) equilibria, and use this characterization to study how the severity of the lemons problem affects the patterns of trade over time.

There is also a large literature that studies the lemons problem in a dynamic setting in which trade is conducted through competitive markets. Most similar to our paper is Janssen and Roy (2002), who also focus on non-stationary equilibria and the patterns of trade over time. In their model, because of free entry, the market price at each date is the expected value of the asset to buyers, so that buyers are somewhat passive and receive zero payoffs in equilibrium. In contrast, the buyers in our model are quite active, and the trade-off they face between current and future payoffs is a dominant feature of the equilibrium characterization.

Our work is also related to the growing literature studying the effects of intervention in frozen markets. Perhaps most similar is Chiu and Koeppl (2009), who introduce asymmetric

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6A parallel literature studies dynamic, decentralized markets with asymmetric information about private values; most related to our work is Moreno and Wooders (2002), who focus on non-stationary equilibria.

7Within the context of a stationary environment, there are many papers in this literature that study how introducing additional institutions, technologies, or contracts can further ease the lemons problem; see, for example, Hendel et al. (2005) and the references therein.
information into the random-matching framework of Duffie et al. (2005) and characterize steady-state equilibria in which the lemons problem is severe enough to shut down trade. They, too, analyze the effect of policy intervention on trading dynamics, and show that a government purchase of low quality assets can help to restore liquidity. We discuss the crucial differences between this result and our own in Section 6.8

From a technical point of view, our work is related to the literature on sequential bargaining with asymmetric information and interdependent values. This literature typically studies the case of a single seller and a single buyer who bargain over time, or a single long-lived seller who faces a sequence of short-lived buyers.9 As in our framework, a feature of these models is that buyers use time to screen different types of sellers.10 However, these models typically have a unique equilibrium, whereas we find multiple equilibria. In Section 5, we discuss how the multiplicity in our environment is driven by the fact that we have a market setting with forward-looking agents.

Lastly, this paper adds to the class of models that provide a theory of endogenous market liquidity based on asymmetrically informed counterparties. On the one hand, there are a variety of search-based models in which information frictions interfere with exchange and thus decrease liquidity; for recent contributions, see Lester et al. (2011), Rocheteau (2011), and the references therein. Alternatively, Eisfeldt (2004) develops a formal relationship between the severity of the lemons problem and liquidity within a competitive market framework. Finally, the dominant theory of liquidity in the finance literature, pioneered by Glosten and Milgrom (1985) and Kyle (1985), also uses informational asymmetries to generate differences in liquidity by focusing on the problem of a market-maker and treating the size of the bid-ask spread as a measure of liquidity.

8Other recent papers studying the effects of asymmetric information on asset market liquidity and policy interventions include Guerrieri and Shimer (2011), Chari et al. (2010), Tirole (2011), Philippon and Skreta (2010), and House and Masatlioglu (2010).

9See Vincent (1989), Evans (1989), and Deneckere and Liang (2006) for examples of the former type of model, and Swinkels (1999), Hörner and Vieille (2009), and Daley and Green (2010) for examples of the latter type of model.

10This basic idea goes back to, at least, Wilson (1980).
2 The Environment

Time is discrete and begins in period $t = 0$. There is an equal mass of infinitely lived buyers and sellers. At $t = 0$, each seller possesses a single, indivisible asset, which is either of high ($H$) or low ($L$) quality. We refer to a seller with a type $j \in \{L, H\}$ asset as a type $j$ seller. The fraction of sellers with a high quality asset at $t = 0$ is $q_0 \in (0, 1)$. We describe below the payoffs to a buyer and a seller from each type of asset.

In every period, each agent’s discount factor $\delta$ is drawn from a continuous and strictly increasing c.d.f. $F$ with support $[\delta, \bar{\delta}]$, where $0 \leq \delta < \bar{\delta} < 1$. These draws are i.i.d. across both agents and time. This is meant to capture the idea that buyers and sellers have different needs at different times. At a given time, some sellers may need to sell their asset more urgently than others, while similarly some buyers may desire immediate consumption more than others. Across time, each individual agent may be more or less patient in any given period.\footnote{Note that all types of agents draw their discount factors from the same c.d.f. $F$. Though non–essential, we think this is reasonable. For a deeper look at the use of random discount factors, see Higashi et al. (2009).} From a technical point of view, this assumption is convenient in that it allows us to focus on pure strategy equilibria; when all buyers share the same discount factor, in general the equilibrium characterization involves mixed strategies (see, e.g., Blouin (2003)).

Preferences

When an asset of quality $j \in \{L, H\}$ is transferred from a seller to a buyer, the buyer obtains utility $u_j$ while the seller suffers disutility $c_j$. This disutility cost can be interpreted as either a direct production cost or as an opportunity cost. In the working paper version, Lester and Camargo (2011), we assume that an asset yields flow payoffs to the seller in every period until it is sold, so that we derive $c_j$ as an opportunity cost explicitly; this alternative specification has no effect on any of the substantive results below. We normalize $c_L$ to 0, and assume that

$$u_H > c_H > u_L > c_L = 0.$$  \hspace{1cm} (1)

The assumptions that $u_H > c_H$ and $u_L > 0$ imply that there are gains from trade in every match. The assumption that $c_H > u_L$ implies that the price that buyers are willing to pay
for a low quality asset would not be accepted by a high quality seller, which is a necessary condition to generate the lemons problem. We also assume that

$$u_H - c_H > u_L,$$

so that gains from trade in a match with a high quality seller are greater than in a match with a low quality seller, which seems to be the natural assumption.

One aspect of our specification of preferences that warrants discussion is that, as in Duffie et al. (2005), buyers and sellers receive different levels of utility from holding a particular asset. This can arise for a multitude of reasons: for example, agents can have different levels of risk aversion, financing costs, regulatory requirements, or hedging needs. In addition, the correlation of endowments with asset returns may differ across agents. The current formulation is a reduced–form representation of such differences.\(^\text{12}\)

**Matching and Trade**

In every period, after the agents draw their discount factors, buyers and sellers are randomly and anonymously matched in pairs. Discount factors and the quality of the seller’s asset are private information. Once matched, the buyer can offer one of two prices, which are fixed exogenously: a high price \(p_h\) that we assume lies in the interval \((c_H, u_H)\), or a low price \(p_L\) that we assume lies in the interval \((0, u_L)\).\(^\text{13}\) The seller can accept or reject. If a seller accepts, trade ensues and the pair exits the market; there is no entry by additional buyers and sellers. If a seller rejects, no trade occurs and the pair remains in the market. This ensures that there is always an equal measure of buyers and sellers in the market.

We impose three restrictions on the exogenous parameters:

$$u_H - p_h > u_L - p_L;$$

\(^3\)

$$\Delta p_h < p_L;$$

\(^4\)

$$\Delta(u_H - p_h) \leq u_L - p_L.$$  

\(^5\)

\(^\text{12}\)For more discussion and examples in which these differences arise endogenously, see, e.g., Duffie et al. (2007), Vayanos and Weill (2008), and Gärleanu (2009).

\(^\text{13}\)One could imagine that buyers possess two indivisible objects that are worth \(p_h\) and \(p_L\) to sellers.
The first assumption implies that, in a world with no information frictions, a buyer would prefer a high quality asset to a low quality asset given the terms of trade. This is again a natural assumption, as it also implies that buyers have no reason to offer $p_\ell$ when the fraction of high quality assets is close to one.

The second and third assumptions allow us to concentrate on the most relevant cases and keep the exposition as clear as possible. In particular, (4) implies that, with strictly positive probability, a type $L$ seller will be sufficiently impatient to accept an offer of $p_\ell$. This is a fairly weak assumption; since we impose no additional structure on the shape of the distribution $F$, this probability can be made arbitrarily small without affecting any of the results below. Finally, since we restrict buyers to offer either $p_\ell$ or $p_h$, we want to focus our attention on the region of the parameter space in which they would never prefer to simply not make an offer at all; (5) is a sufficient condition for this to be true. We stress that, while they simplify the analysis, neither of these assumptions are necessary for many of the results below; for example, all of the basic properties of equilibria derived in Section 3 are robust to relaxing (4) and (5).\footnote{Indeed, in the Supplemental Appendix, we establish all of these properties in an environment with no restrictions on the distribution of discount factors.}

We return to this discussion in greater detail in Section 7, when we discuss the restriction to two prices more generally.

**Strategies and Equilibrium**

A history for a buyer is the set of all of his past discount factors and (rejected) price offers. However, a buyer has no reason to condition behavior on his history: this history is private information, discount factors are i.i.d., and the probability that he meets his current trading partner in the future is zero, as there is a continuum of agents. Moreover, since there is no aggregate uncertainty, the buyer’s history of past offers is not helpful in learning any information about the aggregate state. Thus, a pure strategy for a buyer is a sequence $\mathbf{p} = \{p_t\}_{t=0}^\infty$, with $p_t : [\delta, \bar{\delta}] \rightarrow \{p_\ell, p_h\}$ measurable for all $t \geq 0$, such that $p_t(\delta)$ is the buyer’s offer in period $t$, conditional on still being in the market and drawing discount factor $\delta$.

A history for a seller is the set of all of his past discount factors and all price offers that
he has rejected. The same argument as in the previous paragraph implies that a seller has no reason to condition behavior on his history. Thus, a pure strategy for a type \( j \) seller is a sequence \( a_j = \{a^t_j\}_{t=0}^\infty \), with \( a^t_j : [\delta, \delta] \times \{p_L, p_H\} \to \{0, 1\} \) measurable for all \( t \geq 0 \), such that \( a^t_j(\delta, p) \) is the seller’s acceptance decision in period \( t \), conditional on still being in the market, drawing discount factor \( \delta \), and receiving offer \( p \). We let \( a^t_j(\delta, p) = 0 \) denote the seller’s decision to reject and \( a^t_j(\delta, p) = 1 \) denote the seller’s decision to accept.

We consider symmetric pure-strategy equilibria, which can be described by a list \( \sigma = (p, a_L, a_H) \).\(^{15}\) In order to define equilibria, we must determine payoffs at each date \( t \) under any strategy profile \( \sigma \). Though this is a standard calculation when there is a positive measure of agents remaining in the market, we must also specify what happens when there is a zero measure of agents remaining on each side of the market. More specifically, when all remaining agents trade and exit the market in the current period, we must specify the (expected) payoff to an individual should he choose a strategy that results in not trading.

In order to avoid imposing ad hoc assumptions, we adopt the following procedure for computing these payoffs. Consider the slightly more general version of our model in which, in each period \( t \), a fraction \( \alpha \in (0, 1] \) of the buyers and sellers in the market are matched in pairs, and the remainder do not get the opportunity to trade. The definition of strategies when \( \alpha \in (0, 1) \) is the same as the special case we analyze with \( \alpha = 1 \).\(^{16}\) However, when \( \alpha \in (0, 1) \), in every period \( t \) there is a strictly positive mass of agents remaining in the market, and thus payoffs are always well-defined. We define payoffs when \( \alpha = 1 \) as the limit as \( \alpha \) converges to 1 of payoffs when \( \alpha < 1 \).

More precisely, given a strategy profile \( \sigma \) for all other agents, let \( V^j_t(a|\sigma, \alpha) \) be the expected lifetime payoff to a type \( j \) seller in the market in period \( t \) following the strategy \( a \) and \( V^B_t(p|\sigma, \alpha) \) be the same payoff to a buyer in the market in period \( t \) following the strategy \( p \) when the probability of trade in each period is \( \alpha \in (0, 1) \). Both payoffs are computed before

\(^{15}\)The restriction to pure-strategy equilibria is without loss of generality; with a continuum of agents, any mixed-strategy equilibrium is outcome equivalent to an asymmetric pure-strategy equilibrium.

\(^{16}\)Now a player’s strategy at time \( t \) is conditional on being matched. Moreover, a history for a player also includes the periods in which he was able to trade; for the same reasons given above, a player has no incentive to condition his behavior on this information, though.
discount factors are determined in period \( t \). The payoff to a type \( j \) seller in the market in period \( t \) following the strategy \( a \) is then given by

\[
V_t^j(a|\sigma) = \lim_{\alpha \to 1} V_t^j(a|\sigma, \alpha),
\]

while the payoff to a buyer in the market in period \( t \) following the strategy \( p \) is

\[
V_t^B(p|\sigma) = \lim_{\alpha \to 1} V_t^B(p|\sigma, \alpha).
\]

See the Appendix for the construction of \( V_t^j(a|\sigma, \alpha) \) and \( V_t^B(p|\sigma, \alpha) \) and a proof that the limits (6) and (7) are well–defined regardless of \( a, p, \) and \( \sigma \).

For any strategy profile \( \sigma = (p, a_L, a_H) \), let

\[
A_t^j(p|\sigma) = \int a_t^j(\delta, p)dF(\delta).
\]

By construction, \( A_t^j(p|\sigma) \) is the probability that a seller of type \( j \) in the market in period \( t \) accepts an offer \( p \in \{p_t, p_h\} \). Now let \( T(\sigma) \) be the period in which the market “clears,” i.e., the period in which all sellers remaining in the market accept the price offers made by the buyers; we set \( T(\sigma) = \infty \) if the market never clears. Moreover, let \( q_t(\sigma) \) be the fraction of type \( H \) sellers in the market in period \( t \in \{0, \ldots, T(\sigma)\} \). Notice that \( \{q_t(\sigma)\}_{t=0}^{T(\sigma)} \) satisfies the following law of motion, given \( q_0(\sigma) = q_0 \):

\[
q_{t+1}(\sigma) = \frac{q_t(\sigma) \left\{ 1 - \int A_t^H(p_t(\delta)|\sigma)dF(\delta) \right\}}{q_t(\sigma) \left\{ 1 - \int A_t^H(p_t(\delta)|\sigma)dF(\delta) \right\} + [1 - q_t(\sigma)] \left\{ 1 - \int A_t^L(p_t(\delta)|\sigma)dF(\delta) \right\}}, \tag{8}
\]

Finally, let \( V_t^B(\sigma) \) and \( V_t^j(\sigma) \) denote the payoffs to buyers and type \( j \) sellers, respectively, when they choose the strategy specified by \( \sigma \). By definition, if \( \sigma = (p, a_L, a_H) \), then \( V_t^B(\sigma) = V_t^B(p|\sigma) \) and \( V_t^j(\sigma) = V_t^j(a_j|\sigma) \).

**Definition 1.** The strategy profile \( \sigma^* = (\{p_t^*\}, \{a_t^{L*}\}, \{a_t^{H*}\}) \), along with the law of motion \( \{q_t^*(\sigma)\}_{t=0}^{T(\sigma^*)} \), is an equilibrium if for each \( t \in \{0, \ldots, T(\sigma^*)\} \) and \( j \in \{L, H\} \), we have that:

(i) for all \( \delta \in [\delta, \bar{\delta}] \), \( p_t^*(\delta) \) maximizes

\[
q_t^* \left\{ A_t^H(p|\sigma^*)[u_H - p] + (1 - A_t^H(p|\sigma^*)) \delta V_{t+1}^B(\sigma^*) \right\}
\]

\[
+ (1 - q_t^*) \left\{ A_t^L(p|\sigma^*)[u_L - p] + (1 - A_t^L(p|\sigma^*)) \delta V_{t+1}^B(\sigma^*) \right\};
\]

(ii) \( q_t^* \) satisfies (8) with \( \sigma = \sigma^* \).

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(ii) for each $p \in \{p_\ell, p_h\}$ and $\delta \in [\underline{\delta}, \overline{\delta}]$, $a_{ij}^*(\delta, p) = 1$ if, and only if,

$$p - c_j \geq \delta V_{t+1}^j(\sigma^*);$$

(iii) $q_t^* = q_t(\sigma^*)$.

In words, the strategy profile $\sigma^*$, together with a law of motion $\{q_t^*\}_{t=0}^{T(\sigma^*)}$, is an equilibrium if the behavior of buyers and sellers is optimal in every period $t \leq T(\sigma^*)$ given $\{q_t^*\}$, and $\{q_t^*\}$ is consistent with the aggregate behavior of buyers and sellers. Indeed, the term in (i) is the expected payoff to a buyer in the market in period $t$ when his discount factor is $\delta$ and he offers $p \in \{p_\ell, p_h\}$: conditional on being matched to a type $j$ seller, an offer of $p$ is accepted with probability $A_j^t(p|\sigma^*)$, in which case the buyer’s payoff is $u_j - p$, and rejected with probability $1 - A_j^t(p|\sigma^*)$, in which case the buyer’s payoff is $\delta V_{t+1}^B(\sigma^*)$. Likewise, the optimal behavior for a seller in the market in period $t$ is to accept an offer of $p$ if, and only if, this offer is at least as high as the payoff from continuing to the following period. For ease of exposition, in what we follows we say that $\sigma^*$ is an equilibrium if the pair $(\sigma^*, \{q_t^*\}_{t=0}^{T(\sigma^*)})$ is an equilibrium, where $\{q_t^*\}_{t=0}^{T(\sigma^*)}$ is the law of motion implied by $\sigma^*$.

## 3 Basic Properties of Equilibria

In this section, we establish several properties of equilibria that are helpful in Section 4, where we provide a complete characterization of the equilibrium set. In particular, we establish that the market clears in finite time in every equilibrium and that the fraction of type $H$ sellers in the population is increasing over time before the market clears. It is worth noting that all of these properties are also true in an environment where buyers are free to offer any price; see the Supplemental Appendix for proofs. We start with the following result.

**Lemma 1.** Suppose the market has not cleared before period $t$ and that the fraction of type $H$ sellers in the market is positive. The market clears in period $t$ if, and only if, all buyers in the market offer $p_h$.

Suppose a positive fraction of buyers offer $p_\ell$. Since matching is random, some of them will be matched with type $H$ sellers, who always reject an offer of $p_\ell$ given that $p_\ell < u_L < c_H$;
i.e., \( a_t^H(\delta, p_\ell) = 0 \) for all \( t \geq 0 \) and \( \delta \in [\delta, \bar{\delta}] \). Hence, a positive fraction of buyers who offer \( p_\ell \) have their offer rejected, and thus the market does not clear. Now observe that, since \( p_h \) is the highest price offer a seller can receive, we have \( V_t^j(\sigma) \leq p_h - c_j \) for any strategy profile \( \sigma \). Thus, since \( \bar{\delta} < 1 \), all sellers accept an offer of \( p_h \) in equilibrium; i.e., \( a_t^j(\delta, p_h) = 1 \) for all \( t \geq 0 \), \( \delta \in [\delta, \bar{\delta}] \), and \( j \in \{L, H\} \). Notice that the optimal behavior of type \( H \) sellers is trivial: they accept offers of \( p_h \) and reject offers of \( p_\ell \). Hence, in the analysis below, we will simply take this behavior as given and focus on the behavior of buyers and type \( L \) sellers.

By Lemma 1, for any equilibrium \( \sigma^* \), the market clears in the first period \( T = T(\sigma^*) \) in which all remaining buyers in the market offer \( p_h \). For all \( t < T \), a positive mass of buyers offer \( p_\ell \) and the fraction of type \( L \) sellers who accept this offer is \( F(p_\ell/V_{t+1}^L(\sigma^*)) \). Therefore, the law of motion (8) for the fraction of type \( H \) sellers in the market reduces to

\[
q_{t+1} = \frac{q_t}{q_t + (1 - q_t)[1 - F(p_\ell/V_{t+1}^L(\sigma^*))]}.
\]

Now notice that \( V_{t+1}^L(\sigma^*) \leq p_h \) implies that for all \( t < T \), the fraction of type \( L \) sellers who accept an offer of \( p_\ell \) is at least \( F(p_\ell/p_h) \), and thus bounded away from zero. Hence, the equilibrium behavior of type \( L \) sellers is distinct from that of the type \( H \) sellers. Looking at (9), the following result follows immediately given that \( q_0 \in (0, 1) \).

**Lemma 2.** For any equilibrium \( \sigma^* \), the sequence \( \{q_t\}_{t=0}^{T(\sigma^*)} \) is increasing.

This result is a common feature of dynamic models of trade with adverse selection: since the cost of selling a high quality asset is larger than that of selling a low quality asset, type \( H \) sellers are de facto more patient and remain in the market, on average, longer than type \( L \) sellers. As a result, over time the average quality of assets in the market increases. As we now show, this implies that the market eventually clears in every equilibrium.

**Lemma 3.** In any equilibrium, the market clears in finite time.

The proof of lemma 3 is in the Appendix. Intuitively, if the market never clears, then it must be that the mass of buyers who offer \( p_\ell \) is strictly positive in every period \( t \). Moreover, since \( q_t \) is increasing, the sequence \( \{q_t\}_{t=0}^{\infty} \) must converge to some \( q_\infty \leq 1 \). Given that buyers
discount the future ($\delta < 1$), it must be that $q_\infty < 1$; otherwise, as $q_\infty$ gets sufficiently close to one, the gain from waiting for the market to improve vanishes, and it becomes a strictly dominant strategy for all buyers to offer $p_h$. However, if $q_\infty < 1$, the law of motion (9) implies that the behavior of type $L$ sellers converges to that of type $H$ sellers in the long–run, which is not possible.

4 Characterizing Equilibria

In this section, we provide a complete characterization of the equilibrium set. The first step consists of characterizing the equilibria in which the market clears in the first period of trade, i.e., all buyers offer $p_h$ in $t = 0$. We refer to such equilibria as “0–step” equilibria; more generally, we refer to equilibria in which the market clears in period $k$ as “$k$–step” equilibria. Then we construct the set of 1–step equilibria, recognizing that such equilibria must have the following properties: (i) some agents offer $p_\ell$ at $t = 0$; and (ii) behavior after the first period of trade is given by a 0–step equilibrium. We then repeat this process for $k = 2$, and so on. Since the market clears in finite time in any equilibrium, this recursive procedure exhausts the equilibrium set. All proofs in this section are relegated to the Appendix.

Zero–step equilibria

Denote by $\pi_i^B(q, \delta, v_L, v_H, v_B)$ the payoff to a buyer who offers $p_i$, with $i \in \{\ell, h\}$, when: (i) the fraction of type $H$ sellers in the market is $q \in (0, 1)$; (ii) the buyer’s discount factor is $\delta$; (iii) the continuation payoff to a seller of type $j$ who chooses not to trade is $v_j$; and (iv) the continuation payoff to the buyer should he not trade is $v_B$. Since sellers always accept an offer of $p_h$, we have that

$$\pi_h^B(q, \delta, v_L, v_H, v_B) \equiv \pi_h^B(q) = q(u_H - p_h) + (1 - q)(u_L - p_h).$$

We also know that a type $H$ seller always rejects an offer of $p_\ell$. Therefore,

$$\pi_\ell^B(q, \delta, v_L, v_H, v_B) \equiv \pi_\ell^B(q, \delta, v_L, v_B)$$

$$= (1 - q)F\left(\frac{p_\ell}{v_L}\right)[u_L - p_\ell] + \left\{q + (1 - q)\left[1 - F\left(\frac{p_\ell}{v_L}\right)\right]\right\} \delta v_B, \quad (10)$$
where \( F(p_L/v_L) \) is the fraction of type \( L \) sellers who accept \( p_L \). Since a buyer can always offer \( p_L \) and trade at least with probability \( F(p_L/p_h) > 0 \), it follows that \( v_B > 0 \). Moreover, (5) implies that \( \delta v_B \leq u_L - p_L \). Therefore, it is easy to show that \( \pi^B_t \) is strictly increasing in both \( v_B \) and \( \delta \), and non–increasing in \( v_L \).

Let \( v_B^0(q_0) \) and \( v_L^0(q_0) \) be the payoffs to buyers and type \( L \) sellers in a 0–step equilibrium, respectively.\(^{17}\) It is easy to see that \( v_B^0(q_0) = \pi^B_h(q_0) \) and \( v_L^0(q_0) \equiv v_L^0 = p_h \). To construct the set of 0–step equilibria, consider the strategy profile \( \sigma^0 \) in which, in every \( t \geq 0 \), \( p_t(\delta) = p_h \) for all \( \delta \in [\delta, \bar{\delta}] \) and type \( L \) sellers accept an offer \( p \) if, and only if, \( \delta \leq p/p_h \). It follows from our refinement for computing payoffs when the mass of agents in the market is zero that \( V_B^t(\sigma^0) = v_B^0(q_0) \) and \( V_L^t(\sigma^0) = v_L^0 \) for all \( t \geq 1 \). Indeed, under \( \sigma^0 \), when the fraction of buyers and sellers who are matched in each period is \( \alpha < 1 \), all buyers who get the opportunity to trade exit the market, and so the fraction of type \( H \) sellers among the sellers who remain in the market stays the same. Hence, the strategy profile \( \sigma^0 \) is an equilibrium only if \( v_B^0(q_0) > 0 \) and all buyers find it optimal to offer \( p_h \) in \( t = 0 \), which is true if, and only if,

\[
\pi^B_h(q_0) \geq \pi^B_t(q_0, \delta, v_L^0, v_B^0(q_0)) \tag{11}
\]

since \( \pi^B_t(q_0, \delta, v_L^0, v_B^0(q_0)) \) is strictly increasing in \( \delta \).

In the proof of Proposition 1, we show that there exists a unique \( q^0 \in (0, 1) \) such that (11) is satisfied if, and only if, \( q_0 \geq q^0 \). Moreover, we show that \( v_B^0(q^0) > 0 \), and so \( v_B^0(q_0) > 0 \) for all \( q_0 \geq q^0 \). Thus, \( \sigma^0 \) is an equilibrium if, and only if, \( q_0 \in [q^0, 1) \). Finally, we also show that (11) is the loosest possible constraint on \( q^0 \) that ensures that a buyer finds it optimal to offer \( p_h \) at \( t = 0 \) when all other buyers in the market offer \( p_h \) as well. In other words, no strategy profile \( \tilde{\sigma}^0 \) such that all buyers offer \( p_h \) in \( t = 0 \) is an equilibrium when (11) is violated.

**Proposition 1.** Let \( q^0 \in (0, 1) \) denote the unique value of \( q_0 \) satisfying (11) with equality. There exists a 0–step equilibrium if, and only if, \( q_0 \geq q^0 \).

Notice that \( q_0u_H + (1 - q_0)u_L \geq p_h > c_H \) for any \( q_0 \) in the interval \([q^0, 1)\), so that \( p_h \)

\(^{17}\)In general, we will adopt the convention that a numerical subscript refers to a particular time period, while a numerical superscript refers to the number of periods it takes for the market to clear in equilibrium. In addition, we will use lower case \( v \) to denote equilibrium payoffs.
corresponds to a market–clearing price in a competitive equilibrium. Thus, when the lemons problem is relatively small, i.e., when \( q_0 \) is sufficiently large, the equilibrium outcome in this dynamic, decentralized market coincides with that of a static, frictionless market: trade occurs instantaneously at a single market–clearing price. We will now show, however, that as the lemons problem becomes more severe, equilibrium outcomes no longer resemble those of a centralized competitive market. Instead, these outcomes appear more consistent with models of decentralized trade with search frictions, in the sense that it takes time for buyers and sellers to trade, and they do so at potentially different prices.

**One–step equilibria**

To characterize the set of 1–step equilibria, the following convention will be useful: for any strategy profile \( \sigma \), let \( \sigma_+ \) be the strategy profile such that for all \( t \geq 0 \), the agents’ behavior in period \( t \) is given by their behavior in period \( t + 1 \) under \( \sigma \). In addition, for \( q \in (0, 1) \), let

\[
q^+(q, v_L) = \frac{q}{q + (1 - q)[1 - F(p_\ell/v_L)]}. \tag{12}
\]

By construction, \( q^+(q, v_L) \) is the fraction of type \( H \) sellers in the market in the next period if this fraction is \( q \) in the current period, a positive mass of buyers offer \( p_\ell \), and the continuation payoff to a type \( L \) seller in case he rejects a price offer is \( v_L \). Since \( v_L \leq p_h \), we have that \( F(p_\ell/v_L) \geq F(p_\ell/p_h) > 0 \), and so \( q^+(q, v_L) > q \) for all \( q \in (0, 1) \). Also note that \( q^+(q, v_L) \) is strictly increasing in \( q \) if \( p_\ell/v_L < \delta \) and that \( q^+(q, v_L) \equiv 1 \) if \( p_\ell/v_L \geq \delta \).

Consider a strategy profile \( \sigma^1 \) such that a positive mass of buyers offer \( p_\ell \) in \( t = 0 \) and all buyers offer \( p_h \) in \( t = 1 \). In order for \( \sigma^1 \) to be an equilibrium, it must be that: (i) \( \sigma^1_+ \) is a 0–step equilibrium when the initial fraction of type \( H \) sellers is \( q' = q^+(q_0, v_L^0) \); and (ii) a positive mass of buyers find it optimal to offer \( p_\ell \) in \( t = 0 \) when the market clears in \( t = 1 \).

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18It must also be the case that the type \( j \) sellers accept an offer of \( p \) in \( t = 0 \) if, and only if, \( \delta \leq (p - c_j)/p_h \). This optimal behavior of sellers will be implicitly assumed throughout the analysis.
Formally, the following conditions are necessary and sufficient for $\sigma^1$ to be an equilibrium:

$$q^+(q_0, v_0^L) = q';$$

(13)

$$q' \geq q^0;$$

(14)

$$\pi^B_h(q_0) < \pi^B_\ell(q_0, \bar{\delta}, v_0^L, v_0^B(q')).$$  

(15)

The first condition is simply the law of motion for $q_t$ from $t = 0$ to $t = 1$. Since $v_0^L = p_h$ is a constant, $q^+(q_0, v_0^L)$ is a continuous and strictly increasing function of $q_0$ specifying the unique implied value of $q_1$ in a candidate 1–step equilibrium. The second condition follows from Proposition 1. It ensures that the fraction of type $H$ sellers in $t = 1$ falls in the region of 0–step equilibria. The third condition ensures that a positive mass of buyers find it optimal to offer $p_\ell$ in $t = 0$, given the strategy profile $\sigma^1$. Since $q' \geq q^0$ implies that $v^0_\ell(q') > 0$, $\pi^B_\ell(q_0, \bar{\delta}, v_0^L, v_0^B(q'))$ is strictly increasing in $\bar{\delta}$. Thus, the incentive of a buyer to offer $p_\ell$ in $t = 0$ when the market clears in $t = 1$ increases with the buyer’s patience. As it turns out, (13) and (14) provide a lower bound on the values of $q_0$ for which a 1–step equilibrium exists, while (15) provides an upper bound. Proposition 2 below formalizes these results.

**Proposition 2.** Let $\bar{q}^1$ denote the unique value of $q_0$ satisfying (15) with equality, and define $\underline{q}^1$ to be such that $q^+(\underline{q}^1, v_0^L) = q^0$ if $p_\ell/v_0^L < \bar{\delta}$ and $q^1 = 0$ otherwise. Then $\underline{q}^1 < q^0 < \bar{q}^1 < 1$ and there exists a 1–step equilibrium if, and only if, $q_0 \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^1, \bar{q}^1) \cap (0, 1)$, there exists a unique $q' \in [q_0, 1]$ such that $q'$ is the value of $q_1$ in any 1–step equilibrium when the initial fraction of type $H$ sellers is $q_0$.

In words, if $q_0 = \bar{q}^1$, then the most patient buyer is exactly indifferent between offering $p_\ell$ and $p_h$ when a positive mass of other buyers are offering $p_\ell$. For any $q_0 > \bar{q}^1$, the payoff to such a buyer from immediately trading at price $p_h$ is greater than the payoff from offering $p_\ell$ and not trading with positive probability, in which case the buyer trades at price $p_h$ in the ensuing period (when the fraction of type $H$ sellers in the market is larger). When $p_\ell/v_0^L < \bar{\delta}$, $\bar{q}^1$ is the unique value of $q_0$ such that, if a positive mass of buyers offer $p_\ell$, then the fraction of high quality sellers in the next period is $q^0$, the minimum value required for market clearing; notice that $\bar{q}^1 > 0$ in this case. If even the most patient type $L$ seller would rather accept an offer of $p_\ell$ today than wait one period for an offer of $p_h$, i.e., if $p_\ell/v_0^L \geq \bar{\delta}$, then $\underline{q}^1 = 0$. 

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The fact that $q^0 < q^1$ implies that there are both 0–step and 1–step equilibria when $q_0 \in [q^0, q^1)$. In this region, if all other buyers are offering $p_h$, then the payoffs to trading at $t = 0$ and at $t = 1$ are the same, and so it is optimal for an individual buyer to offer $p_h$ no matter his discount factor. However, if a positive mass of other buyers are offering $p_\ell$, then the market does not clear at $t = 0$ and the payoff to trading at $t = 1$ increases (since $q_1 > q_0$), rendering it optimal for patient buyers to offer $p_\ell$ and incur a chance that they trade only in the next period. We continue this discussion of multiple equilibria in Section 5.

Let $Q^1_+(q_0) \equiv q^+(q_0, v^0_L)$ denote the value of $q_1$ corresponding to an initial value $q_0$ in a 1–step equilibrium. Then the payoff to a buyer in a 1–step equilibrium is

$$v^1_B(q_0) = \int \max\{\pi^B_h(q_0), \pi^B_\ell(q_0, \delta, v^0_L, v^0_B(Q^1_+(q_0)))\} \, dF(\delta).$$

The payoff to a type $L$ seller in a 1–step equilibrium is

$$v^1_L(q_0) = \xi^1(q_0)p_h + (1 - \xi^1(q_0)) \int \max\{p_\ell, \delta v^0_L\} \, dF(\delta),$$

where

$$\xi^1(q_0) = \int \mathbb{I}\{\pi^B_h(q) \geq \pi^B_\ell(q, \delta, v^0_L, v^0_B(Q^1_+(q_0)))\} \, dF(\delta)$$

denotes the fraction of buyers that offer $p_h$ at $t = 0$ and $\mathbb{I}$ denotes the indicator function. It turns out that $\xi^1$ is continuous and increasing in $q_0$, with $\lim_{q_0 \to q^1} \xi^1(q_0) = 1$, which implies that $v^1_L$ is also continuous and increasing in $q_0$, with $\lim_{q_0 \to q^1} v^1_L(q_0) = v^0_L$. Lemma 4 in the Appendix establishes these results formally, as well as some additional properties of $v^1_B$ that are useful in constructing 2–step equilibria. In what follows, we let $v^1_L(q^1) = \lim_{q_0 \to q^1} v^1_L(q_0)$.

**Two–step equilibria**

We now provide a complete characterization of 2–step equilibria. As it turns out, the process of characterizing $k$–step equilibria is nearly identical for all $k \geq 2$. Thus, the methodology developed here will allow for a complete characterization of equilibria in the next subsection.

Consider a strategy profile $\sigma^2$ such that a positive mass of buyers offer $p_\ell$ in $t = 0$ and $t = 1$, and then all buyers offer $p_h$ in $t = 2$. In order for $\sigma^2$ to be an equilibrium, it must
satisfy the following three necessary and sufficient conditions:

\[ q^+ (q_0, \ell(q')) = q'; \quad (16) \]

\[ q' \in [q^1, \bar{q}^1) \cap (0, 1); \quad (17) \]

\[ \pi^B_B(q_0) < \pi^B_B(q_0, \delta, \ell(q'), \ell(q')). \quad (18) \]

The first condition is the analog of condition (13) for 1–step equilibria; it is the law of motion for \( q_t \) from \( t = 0 \) to \( t = 1 \), conditional on a 1–step equilibrium beginning at \( t = 1 \). Unlike (13), the fraction \( q' \) in (16) is the solution to a fixed point problem: if the type \( L \) sellers expect continuation payoffs to be that of a 1–step equilibrium in which the initial fraction of type \( H \) sellers is \( q' \), then the fraction of type \( L \) sellers who accept an offer of \( p_\ell \) in \( t = 0 \) must be such that this conjecture is correct. This fixed point problem does not appear in (13) since the payoff \( \ell_L^0 \) for type \( L \) sellers in a 0–step equilibrium is independent of the initial fraction of high quality assets. The second condition ensures that there exists a 1–step equilibrium at \( t = 1 \) given an initial fraction \( q' \) of high quality assets. The final condition ensures that a positive mass of buyers find it optimal to offer \( p_\ell \) in \( t = 0 \) when \( \sigma^2_+ \) is a 1–step equilibrium.

Since \( \ell_L^0(q') \leq \ell_L^0(q^1) = \ell_L^0 \) for all \( q' \in [q^1, \bar{q}^1) \cap (0, 1) \), we have that \( p_\ell/\ell_L^0 \geq \delta \) implies that \( q^+(q_0, \ell_L^0(q')) = 1 \) for all \( q' \in [q^1, \bar{q}^1) \cap (0, 1) \). Thus, no 2–step equilibrium exists if \( p_\ell/\ell_L^0 \geq \delta \). Intuitively, when \( p_\ell/\ell_L^0 \geq \delta \), all type \( L \) sellers with \( \delta < \delta \) strictly prefer to accept on offer of \( p_\ell \) if continuation payoffs are that of a 1–step equilibrium. Therefore, the fraction of type \( H \) sellers in the market at \( t = 1 \) is one, and the market clears in two periods.

Suppose then that \( p_\ell/\ell_L^0 < \delta \). We show in the proof of Proposition 3 that (16) and (17) imply (18). Intuitively, the incentive of the most patient buyer to choose \( p_\ell \) in \( t = 0 \) is even greater than his incentive to choose \( p_\ell \) in \( t = 1 \), when the fraction of type \( H \) sellers in the market is \( q' > q_0 \). Hence, if the most patient buyer strictly prefers to choose \( p_\ell \) in \( t = 1 \), which is true by (17), then he also strictly prefers to offer \( p_\ell \) at \( t = 0 \) and (18) is satisfied. Therefore, (16) and (17) are necessary and sufficient conditions for a 2–step equilibrium.

Let \( Q^2_+ : q_0 \mapsto q' \) denote the map defined by (16). In words, \( Q^2_+(q_0) \) is the value of \( q_1 \) in a 2–step equilibrium, given \( q_0 \). In the proof of Proposition 3, we show that \( Q^2_+(q_0) \) is a well–defined function that is both continuous and strictly increasing in \( q_0 \). Therefore, for
any \( q_0 \), there is a unique value of \( q_1 \) in any candidate 2–step equilibrium. These properties of \( Q_+^2(q_0) \) greatly simplify the characterization of 2–step equilibria: the necessary and sufficient conditions (16) and (17) become \( Q_+^2(q_0) \geq \delta \) and \( Q_+^2(q_0) < \bar{\delta} \). Hence, the lower bound on \( q_0 \) for which a 2–step equilibrium exists is the value of \( q_0 \) such that \( Q_+^2(q_0) = \bar{\delta} \), while the upper bound is the value of \( q_0 \) such that \( Q_+^2(q_0) = \bar{\delta} \). Proposition 3 below summarizes.

**Proposition 3.** Suppose that \( \bar{\delta} > p_\ell/v_L^0 \). Let \( q^2 \) be the unique solution to \( q_+(q^2, v_L^1(q^1)) = \bar{\delta} \), and define \( \bar{q}^2 \) to be such that \( q_+(q^2, v_L^1(q^1)) = \bar{\delta} \) if \( p_\ell/v_L^1(q^1) < \bar{\delta} \) and \( q^2 = 0 \) otherwise. Then \( q^2 < q^1 < \bar{q}^2 < \bar{\delta} \) and there exists a 2–step equilibrium if, and only if, \( q_0 \in [q^2, \bar{q}^2) \cap (0, 1) \). Moreover, for each \( q_0 \in [q^2, \bar{q}^2) \cap (0, 1) \), there exists a unique \( q' \in [q^1, \bar{\delta}] \) such that \( q' \) is the value of \( q_1 \) in any 2–step equilibrium when the initial fraction of type H sellers is \( q_0 \).

Figure 2 provides some intuition for the equilibrium characterization so far. After deriving \( q^0 \) and \( \bar{\delta} \), we identified \( q^1 \) as the value of \( q_0 \) that would “land” exactly on \( q^0 \) at \( t = 1 \) given the law of motion \( Q_+^1(q_0) \). Since this law of motion is continuous and strictly increasing in \( q_0 \) (for \( \bar{\delta} > p_\ell/v_L^0 \)), we are assured that any \( q_0 > q^1 \) will “land” at \( q' > q^0 \) in a candidate 1–step equilibrium. Moving backwards, we then identified \( q^2 \) and \( \bar{q}^2 \) as the values of \( q_0 \) that would “land” exactly on \( q^1 \) and \( \bar{\delta} \), respectively, given the law of motion \( Q_+^2(q_0) \). Though this law of motion is slightly more complicated, the fact that it remains continuous and strictly increasing assures us that any \( q_0 \in [q^2, \bar{q}^2) \cap (0, 1) \) will “land” within the region of 1–step equilibria. Finally, since \( v_L^1(\bar{\delta}) = v_L^0, \bar{\delta} > q^0 \), and

\[
q_+(\bar{q}^2, v_L^0) = \bar{\delta} = q^0 = q_+(q^1, v_L^0),
\]

the fact that \( \bar{q}^2 > q^1 \) follows immediately from the fact that \( q_+(q_0, v_L) \) is strictly increasing in \( q_0 \) for any \( v_L \) such that \( p_\ell/v_L < \bar{\delta} \). Thus, there is no “gap” in the values of \( q_0 \) for which 1–step and 2–step equilibria exist; rather, the multiplicity of equilibria that emerged in the region \([q^0, q^1)\) also takes place in the region \([q^1, \bar{\delta})\).

To finish the characterization of 2–step equilibria, notice that since \( Q_+^2(q_0) \) is uniquely defined, so too are payoffs for type L sellers and buyers in such an equilibrium; see equations (20) to (22) below to see how to construct the payoffs \( v_L^2(q_0) \) and \( v_B^2(q_0) \), along with the
fraction $\xi^2(q_0)$ of buyers that offer $p_h$ at $t = 0$ in a 2–step equilibrium. As in the case of 1–step equilibria, it turns out that these functions have several properties that are helpful for characterizing the set of 3–step equilibria. Lemma 5 in the Appendix establishes these properties formally.

**A Full Characterization**

The characterization of $k$–step equilibria for $k \geq 3$ proceeds by induction and follows almost exactly the characterization of 2–step equilibria. Hence, for ease of exposition, we just sketch the process here and leave the details for the Appendix.

As the first step, we take as given the range of values of $q_0$ for which a $(k - 1)$–step equilibrium exists, $[q_{k-1}^{-}, q_{k-1}^{+}] \cap (0, 1)$, along with the corresponding equilibrium payoffs for type L sellers, $v_{k-1}^{L}(q_0)$, and buyers, $v_{k-1}^{B}(q_0)$. Then we define the fixed–point mapping $Q_k^+: q_0 \mapsto q'$ as the solution to

$$q^+ (q_0, v_{k-1}^{L}(q')) = q', \quad (19)$$

where $q' \in [q_{k-1}^{-}, q_{k-1}^{+}] \cap (0, 1)$. We establish two important results: first, that the most patient buyer strictly prefers to offer $p_\ell$ at $t = 0$ if (19) is satisfied; and second, that $Q_k^+$ is continuous and strictly increasing. This implies that a $k$–step equilibrium exists if, and only if, $q_0 \in [q_k^{-}, q_k^{+}] \cap (0, 1)$, where the lower bound $q_k^{-}$ is such that $Q_k^+(q_k^{-}) = q_{k-1}^{-}$ if $p_\ell/v_{k-1}^{L}(q_{k-1}^{-}) < \delta$ and $q_k = 0$ otherwise, and the upper bound $q_k^{+}$ satisfies $Q_k^+(q_k^{+}) = q_{k-1}^{+}$. Moreover, we show that $q_k^{-} \leq q_{k-1}^{+} < q_k^{+} < q_{k-1}^{+}$, so that, in particular, there is no gap in the values of $q_0$ for which $k$–step and $(k - 1)$–step equilibria exist.
Theorem 1. There exists \( k \) such that, for all \( k \) there exists if, and only if, 
\[
\pi_h^B(q_0), \pi^B_t(q_0, \delta, v^k_{L-1}(Q^k_+(q_0)), v^k_{B-1}(Q^k_+(q_0))) \]
d itself. The payoff for type \( L \) sellers is
\[
v^k_L(q_0) = \xi^k(q_0)p_h + (1 - \xi^k(q_0)) \max \{ p_\ell, \delta v^k_{L-1}(Q^k_+(q_0)) \} dF(\delta),
\]
where
\[
\xi^k(q_0) = \int \mathbb{1}\{\pi^B_h(q) \geq \pi^B_t(q_0, \delta, v^k_{L-1}(Q^k_+(q_0)), v^k_{B-1}(Q^k_+(q_0)))\} dF(\delta)
\]
is the fraction of buyers who offer \( p_h \) at \( t = 0 \). We begin this induction process with \( k = 3 \), and continue it as long as \( p_\ell/v^k_{L-1}(\bar{q}^{k-1}) < \bar{\delta} \), which ensures \( \bar{q}^k > 0 \) and thus the existence of \( k \)-step equilibria. Theorem 1 provides a full characterization of the equilibrium set.

**Theorem 1.** There exists \( 1 \leq K < \infty \) and sequences \( \{q^k\}_{k=0}^K \) and \( \{\bar{q}^k\}_{k=0}^K \), with \( q^0 = 1 \), \( q^K = 0 \), and \( q^k \leq q^{k-1} < \bar{q}^k < q^{k-1} \) for all \( k \in \{1, \ldots, K\} \), such that a \( k \)-step equilibrium exists if, and only if, \( q_0 \in [q^k, \bar{q}^k) \cap (0, 1) \). Moreover, for each \( q_0 \in [q^k, \bar{q}^k) \cap (0, 1) \), there exists a unique \( q' \in [q^{k-1}, \bar{q}^{k-1}) \) such that \( q' = Q^k_+(q_0) \) is the value of \( q_1 \) in any \( k \)-step equilibrium when the initial fraction of type \( H \) sellers is \( q_0 \).

The payoffs for buyers and type \( L \) sellers are uniquely defined in every equilibrium and are determined recursively as follows: (i) \( v^0_B(q_0) = \pi^B_h(q_0) \) and \( v^0_L(q_0) \equiv p_h; \) (ii) for each \( k \in \{1, \ldots, K\} \), \( v^k_B \) and \( v^k_L \) are given by (20) and (21), respectively.

The cutoffs \( \{q^k\}_{k=0}^{K-1} \) and \( \{\bar{q}^k\}_{k=1}^K \) are defined recursively as follows: (i) \( q^0 \) is the unique value of \( q_0 \) for which \( \pi^B_h(q_0) = \pi^B_t(q_0, \delta, v^0_L, v^0_B(q_0)) \) and, for each \( k \in \{1, \ldots, K\} \), \( q^k \) is such that \( q^+(q^k, v^k_{L-1}(q^k)) = q^{k-1} \) if \( p_\ell/v^k_{L-1}(q^k) < \bar{\delta} \), and \( q^k = 0 \) otherwise; (ii) \( \bar{q}^1 \) is the only value of \( q_0 \) for which \( \pi^B_h(q_0) = \pi^B_t(q_0, \delta, v^0_L, v^0_B(q^+(q_0, v^0_L))) \) and, for each \( k \in \{2, \ldots, K\} \), \( \bar{q}^k \) is such that \( q^+(\bar{q}^k, v^k_{L-1}(\bar{q}^k)) = q^{k-1} \). Finally, \( K = \max\{k : p_\ell/v^k_{L-1}(\bar{q}^{k-1}) < \bar{\delta}\} \).

Theorem 1 specifies a sequence of cutoffs that partition the interval \((0, 1)\) into regions such that, for all \( q_0 \) in one such region, there exists an equilibrium in which the market takes the same number of periods \( k \) to clear. Figure 3 below illustrates these cutoffs for the case
in which \(u_H = 1, p_h = 0.6, c_H = 0.5, u_L = 0.42, p_L = 0.05\), and \(F\) is uniformly distributed over \([0, 0.5]\). In addition to plotting these cutoffs, we have also highlighted the maximum and minimum number of periods it takes before the market clears for each \(q_0 \in (0, 1)\).

Notice that there is a natural monotonicity to the equilibria: for any \(0 < q_0 < q_0' < 1\), if there exists a \(k\)-step equilibrium when the initial fraction of high quality assets is \(q_0\), then there exists a \(k'\)-step equilibrium with \(k' \leq k\) when the initial fraction of high quality assets is \(q_0'\). This is true in general since \(\bar{q}^k\) is strictly decreasing in \(k\) by Theorem 1, and so an increase in \(q_0\) reduces (weakly) the maximum number of periods it takes for the market to clear. Also notice that in the example, the market clears in at most four periods. However, depending on the distribution \(F\) and parameters of the model, market clearing can take a large number of periods when \(q_0\) is small. We show this in the Appendix.

### 5 Discussion

We now illustrate how the theory developed above can provide insight into a number of important issues. First, we study how the initial composition of assets in the market affects the expected amount of time it takes to sell—or the illiquidity of—high quality assets. Then, we study the dynamics of trade for a given value of \(q_0\), exploring the model’s implications for how prices, trading volume, and average quality evolve over time in this type of environment.

Establishing such a benchmark is important, as it not only allows us to understand how frozen
markets can thaw over time on their own, but it also provides a framework to formally analyze the effects of various policies intended to unfreeze such markets; we discuss one particular policy intervention in the next section. Finally, the existence of multiple equilibria for a given value of \( q_0 \) suggests that coordination failures can exacerbate liquidity problems in dynamic, decentralized markets with adverse selection. Since such multiplicity does not arise in several closely related (and well-known) environments, we end this section with a discussion of those features of our framework that are crucial for generating these coordination failures.

**Liquidity and Lemons**

Here we study how the fraction of lemons in the market affects the liquidity of high quality assets.\(^{19}\) An asset is typically considered liquid if it can be sold quickly and at little discount. In many models, trade is instantaneous by construction, and thus the only measure of liquidity is the difference between the actual market price and the price in some frictionless benchmark; in these models, time is a margin that simply cannot adjust.\(^{20}\) In the current model, the opposite is true: since \( p_h \) is the only price that type \( H \) sellers accept, the appropriate measure of liquidity for these assets is the expected amount of time it takes to sell them. We derive this statistic below and use it to study the relationship between the severity of the lemons problem (i.e., the value of \( q_0 \)) and the liquidity of high quality assets.

Consider a \( k \)-step equilibrium with initial fraction \( q_0 \in [\tilde{q}_k^{\text{L}}, \tilde{q}_k^{\text{H}}] \cap (0, 1) \) of high quality assets, and define the sequence \( \{q_t\}_{t=1}^k \) to be such that \( q_t = Q_{t+1}^{k-t+1}(q_{t-1}) \) for all \( t \in \{1, \ldots, k\} \). By construction, \( q_t \) is the fraction of high quality assets in the market in period \( t \). Therefore,

\(^{19}\)Focusing on the ability to sell high quality assets is standard in this literature, going back to Akerlof (1970). Of course, a seller can always sell a low quality asset instantaneously at price \( p_L \).

\(^{20}\)In the finance literature, the typical measure of liquidity is the (inverse of) the bid–ask spread, which can be generated by exogenous transaction costs (see, e.g., Amihud and Mendelson (1986) and Constantinides (1986)), asymmetric information (see Kyle (1985) and Glosten and Milgrom (1985)), or search frictions (see Duffie et al. (2005)), among other things. Eisfeldt (2004) provides an alternative definition of liquidity in a competitive market setting where trade is again instantaneous. In her model, an influx of low quality assets drives down the (pooling) equilibrium price of the high quality asset, thus decreasing a seller’s ability to exchange the latter type of asset for cash.
the probability that a type $H$ seller trades his asset in period $t \in \{0, \ldots, k\}$ is

$$\lambda^k(t|q_0) = \left\{ \prod_{s=0}^{t-1} [1 - \xi^{k-s}(q_s)] \right\} \xi^{k-t}(q_t),$$

where $\xi^k(q)$ is the fraction of buyers who offer $p_h$ in the first period of trade in a $k$–step equilibrium when the starting fraction of type $H$ sellers is $q$. The expected number of periods it takes to sell a high quality asset in the equilibrium under consideration is then

$$E^k_H(q_0) = \sum_{t=0}^{k} \lambda^k(t|q_0)t.$$

We know from the proof of Theorem 1 that, in any $k$–step equilibrium, both $\xi^k(q_0)$ and $Q^k_+(q_0)$ are increasing in $q_0$. Hence, an increase in $q_0$ implies an increase in the fraction of buyers who offer $p_h$ in the first period of trade. Moreover, an increase in $q_0$ also leads to an increase in $q_t$ for all $t \in \{1, \ldots, k\}$, which in turn implies an increase in the fraction of buyers who offer $p_h$ in every period before the market clears. Taken together, these two facts help to establish that $E^k_H(q_0)$ is a decreasing function of $q_0$; we present a formal proof of this result in Lemma 6 in the Appendix.

As we established earlier, for some values of $q_0$ there exist multiple equilibria that take a different number of periods for the market to clear. This, of course, makes comparing the liquidity of high quality assets across different values of $q_0$ difficult. Here we do not take a stance on equilibrium selection and instead focus on the relationship between the minimum expected number of periods it takes for a type $H$ seller to sell his asset and $q_0$. Let $\mathcal{E}(q_0)$ be given by

$$\mathcal{E}(q_0) = \min \left\{ E^k_H(q_0) : \exists a k$–step equilibrium given $q_0 \right\}.$$

In Lemma 7 in the Appendix we use the fact that $E^k_H(q_0)$ is decreasing in $q_0$ to show that $\mathcal{E}(q_0)$ is decreasing in $q_0$. Thus, a reduction in the initial fraction of high quality assets reduces their liquidity in the sense that it increases the smallest expected amount of time it takes to sell them.\(^{21}\)

\(^{21}\)Alternatively, one could compare the liquidity of high quality assets across different values of $q_0$ using a particular equilibrium selection rule. For example, if $k_{\text{max}}(q_0) = \max \{k : \exists a k$–step equilibrium given $q_0 \}$ and $E_{\text{max}}(q_0) = E^{k_{\text{max}}}(q_0)$, then it is possible to show that $E_{\text{max}}(q_0)$ is also decreasing in $q_0$.\(^{21}\)
The Dynamics of Trade

We now illustrate typical market dynamics for a given value of $q_0$. The numerical example of Section 4 is a convenient vehicle for conveying the intuition; we choose $q_0 = 0.1$, which falls within the set of 3–step equilibria. The average price in period $t \in \{0, \ldots, k\}$ in a $k$–step equilibrium is given by

$$p_{t}^{avg} = \xi^{k-t}(q_t) p_h + \left[1 - \xi^{k-t}(q_t)\right] p_\ell,$$

where, as above, $\{q_t\}_{t=1}^{k}$ is the sequence such that $q_t = Q_{s+t+1}^{s+t+1}(q_{t-1})$ for all $t \in \{1, \ldots, k\}$. In figure 4 below, we plot the evolution of $q_t$ and $p_{t}^{avg}$ in the example.

In the first two periods of trade, the fraction of high quality assets is sufficiently low that all buyers offer $p_\ell$. All type $H$ sellers and patient type $L$ sellers reject this offer, but sufficiently impatient type $L$ sellers accept, causing the average quality of assets in the market to be higher in the following period. In the third period of trade, the fraction of high quality assets is sufficiently high that some impatient buyers offer $p_h$, increasing the average price. Still, patient buyers continue to offer $p_\ell$ and (perhaps) wait for market conditions to improve. In the fourth period of trade, all remaining buyers offer $p_h$ and the market clears. Thus, average prices increase over time along with average quality. In the example, the price path exhibits an $S$–shape: prices are persistently low in early periods, and then quickly increase in the latter stages of trade.

In figure 4 below, we plot the evolution of $q_t$ and $p_{t}^{avg}$ in the example.

Many of the features of the above example are true in general. We know from the proof of Theorem 1 that $\xi^s(q) \leq \xi^{s-1}(Q^s_q(q))$ for all $s \in \{0, \ldots, k\}$ in any $k$–step equilibrium. Hence, the fraction of buyers who offer $p_h$ increases over time, so that $p_{t}^{avg}$ increases over time as well. Now note that if

$$\pi^B_h(q_0) \leq (1 - q_0)F \left(\frac{p_\ell}{v_{L}^{k-1}(q_1)}\right) [u_L - p_\ell]$$

in a $k$–step equilibrium, then all buyers prefer to offer $p_\ell$ at $t = 0$, so that only low quality assets are exchanged in the first period of trade. The trade of high quality assets remains frozen until the first period in which buyers with $\delta = \delta$ find it strictly optimal to offer $p_h$. 

28
Multiplicity of Equilibria

The presence of multiple equilibria for some values of $q_0$ suggests that liquidity problems can be exacerbated by coordination failures. At the heart of this multiplicity is the fact that the behavior of an individual buyer depends on the future composition of assets in the market, which in turn is determined by the aggregate behavior of buyers.

Identifying the ingredients of our framework that lead to multiple equilibria—that there are many buyers, and that these buyers are forward-looking—is helpful in understanding why such multiplicity does not typically arise in certain related environments. For example, in models of bargaining with asymmetric information in which there is only one buyer and one seller (see, e.g., Vincent (1989), Evans (1989), and Deneckere and Liang (2006)), clearly there is no scope for coordination between buyers’ actions; as a result, there is typically a unique sequential equilibrium in these models. Alternatively, in similar frameworks in which a single seller with private information meets a sequence of buyers (see, e.g., Hörner and Vieille (2009) and the references therein), the buyers are typically assumed to be myopic. As a result, there is no potential for buyers to coordinate their behavior based on future payoffs, and again the type of multiplicity that we find here does not emerge. Importantly, the emergence of multiple equilibria is not a consequence of our restriction to two prices, as we discuss more formally in section 7.

6 Application: The Market for Legacy Assets

At the core of the recent financial crisis was the frozen market for asset-backed securities, which threatened not only many large financial institutions, but also the myriad individuals and firms who depend on these institutions for funding. The lack of liquidity in this mar-

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22The two ingredients we identify are not sufficient for multiplicity. For example, in Janssen and Roy (2002), there is a continuum of forward-looking buyers and sellers who trade in a sequence of centralized markets in the presence of asymmetric information. However, their equilibrium requires that buyers receive zero expected payoffs from trading at any date, thus precluding the possibility of the multiplicity we find in our model. The two ingredients are also not necessary for multiplicity. Gerardi et al. (2010) show that multiple equilibria arise in sequential bargaining with asymmetric information when the party that makes the offers is the informed one; signalling is the source of multiplicity in their environment.
ket posed a threat to the economy at large. The U.S. Treasury department described the “challenge of legacy assets” as follows:

One major reason [for the prolonged recession] is the problem of “legacy assets”—both real estate loans held directly on the books of banks (“legacy loans”) and securities backed by loan portfolios (“legacy securities”). These assets create uncertainty around the balance sheets of these financial institutions, compromising their ability to raise capital and their willingness to increase lending... As a result, a negative cycle has developed where declining asset prices have triggered further deleveraging, which has in turn led to further price declines. The excessive discounts embedded in some legacy asset prices are now straining the capital of U.S. financial institutions, limiting their ability to lend and increasing the cost of credit throughout the financial system.

Many believe that the root of the failure in the market for asset–backed securities was asymmetric information between the owners of these assets and the traditional buyers. Indeed, the decline of housing prices in various parts of the country introduced considerable heterogeneity into the quality of residential mortgage–backed securities, and many of the usual buyers of these assets did not possess the expertise to comfortably value the assets that were being offered by financial institutions.23 As a result, both the prices and the volume of these assets being sold quickly dropped.24 An important policy question then emerged: how can the government (or the central bank) best intervene to restore liquidity in a market suffering from adverse selection?

Though this question received a lot of attention at the height of the financial crisis, much of the discussion took place outside the realm of formal economic analysis. Our framework, though stylized, shares many features of the market for asset–backed securities, and thus...

23The financial institutions that were selling these assets often had a team of analysts that had purchased the underlying assets (e.g., mortgages), studied their properties, and worked closely with the rating agencies to bundle them into more opaque final products. An extreme example of this asymmetric information is the “Abacus” deal, in which Goldman Sachs created and sold collateralized debt obligations to investors while simultaneously betting against them. In general, there are many reasons to believe that financial institutions often have better information about the quality of their assets than potential buyers, perhaps because they learn about the asset while they own it (as argued by Bolton et al. (2011)), or because they conduct research about the asset in anticipation of selling it (as argued by Guerrieri and Shimer (2011)). By now, there is a large literature on the role of asymmetric information in the financial crisis; see, e.g., Gorton (2009) and the references therein.

24For a detailed analysis, see Krishnamurthy (2010).
provides a parsimonious environment to analyze the efficacy of policy interventions on market liquidity. In this section, we analyze a particular program that was proposed in the U.S. in March of 2009: the Public–Private Investment Program for Legacy Assets.

Under this program, the government issued non–recourse loans to private investors to assist in buying legacy assets, with a minimum fraction of the purchase price being financed by the investor’s own equity. This program essentially subsidizes the buyer’s purchase and partially insures his downside loss; if the asset turns out to be a lemon, the buyer can default and incur only a fraction of the total loss from the purchase (his equity investment). Intuitively, such a program would seem to unambiguously (at least weakly) improve market liquidity. Indeed, in the context of a static or stationary model, any policy that reduces the loss to a buyer of acquiring a lemon should raise equilibrium prices, and thus increase the incentive for high quality sellers to trade. The only question would seem to be about the optimal size of such an intervention.

However, within the context of a non–stationary environment, we show that such a program can potentially slow down a market recovery. The crucial insight is that an intervention of this type potentially increases both current and future payoffs to a low quality seller. If the increase in future payoffs is large relative to the increase in current payoffs, then a low quality seller can in fact become less willing to trade immediately, thus slowing down the thawing process described in our equilibrium characterization. One can show that, in this case, the expected number of periods it takes to sell a high quality asset also increases, so that markets become more illiquid. Since any delay in trade is inefficient in this environment, a program that slows down the thawing process will then, in general, decrease welfare (even without accounting for the cost of the program). This counter–intuitive consequence of the government’s intervention, which we illustrate via a numerical example, highlights that both the size and the timing—and in particular the duration—of a policy are crucial.

For one, as in our model, buyers and sellers in this market negotiate bilaterally, as opposed to trading against their budget constraint in a competitive, centralized market where the law of one price prevails. Moreover, this market is inherently dynamic and non–stationary: there is a relatively fixed stock of assets of a particular vintage, and the manner in which the composition of assets remaining in the market evolves over time affects both prices and the incentive of market participants to trade.

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25 For one, as in our model, buyers and sellers in this market negotiate bilaterally, as opposed to trading against their budget constraint in a competitive, centralized market where the law of one price prevails. Moreover, this market is inherently dynamic and non–stationary: there is a relatively fixed stock of assets of a particular vintage, and the manner in which the composition of assets remaining in the market evolves over time affects both prices and the incentive of market participants to trade.
determinants of its ability to restore market liquidity.

In an attempt to capture this policy response, suppose now that a buyer who pays price \( p \) for an asset can borrow \((1 - \gamma)p\) from the government. For simplicity, assume the buyer observes the quality of the asset immediately after buying it and then faces the following choice: either pay back the loan to the government, or default on the loan and surrender the asset. A buyer who pays price \( p \) for a type \( j \) asset repays his loan if, and only if, \( u_j - (1 - \gamma)p > 0 \). Thus, a buyer who receives a high quality asset always repays his loan, as does a buyer who pays \( p_\ell \) for a low quality asset. However, a buyer who pays \( p_h \) for a low quality asset defaults if \( \gamma \leq 1 - u_L/p_h \). Therefore, this policy amounts to a transfer \( \tau = (1 - \gamma)p_h - u_L \in [0, p_h - u_L] \) to the buyers who pay \( p_h \) for a low quality asset.

Denote the payoff to a buyer from offering \( p_h \) given a transfer \( \tau \) by

\[
\pi_h^B(q_0, \tau) = q_0(u_H - p_h) + (1 - q_0)(u_L - p_h + \tau).
\]

The payoff to a buyer from offering \( p_\ell \) is still given by (10), and the characterization of the equilibrium set proceeds in exactly the same way as in Section 4. In particular, Theorem 1 is still valid with the only difference that now, in the recursive procedure that determines the equilibrium payoffs, the payoff to a buyer in a 0–step equilibrium is \( \pi_h^B(q_0, \tau) \).

Let \( v_L^k(q_0, \tau) \) and \( v_L^k(q_0, \tau) \) be, respectively, the payoffs to buyers and type \( L \) sellers in a \( k \)-step equilibrium when the transfer is \( \tau \). Moreover, let \( q^k(\tau) \) and \( \overline{q}^k(\tau) \) be, respectively, the lower and upper cutoffs for a \( k \)-step equilibrium as a function of \( \tau \). The cutoff \( q^0(\tau) \) is the unique value of \( q_0 \) such that

\[
\pi_h^B(q_0, \tau) = \pi_\ell^B(q_0, \overline{q}, v_0^L, v_{B}^0(q_0, \tau)),
\]

where \( v_0^L = v_0^B(q_0, \tau) \equiv p_h \). The cutoffs \( q^1(\tau) \) and \( \overline{q}^1(\tau) \) are the unique values of \( q_0 \) satisfying the following two equations, respectively:

\[
q^+(q_0, v_0^L) = \overline{q}^0;
\]

\[
\pi_\ell^B(q_0, \overline{q}, v_0^L, v_{B}^0(q_0, \tau), \tau)) = \pi_h^B(q_0, \tau).
\]

It is straightforward to show that \( q^0(\tau), q^1(\tau), \) and \( \overline{q}^1(\tau) \) are decreasing in \( \tau \). Therefore, if the initial fraction of high quality assets is sufficiently large, an increase in \( \tau \) can decrease
the amount of time it takes for the market to clear, and thus increase market liquidity. For example, for $\tau \in (0, p_h - u_L)$, there exists a 0–step equilibrium when $q_0 \in (q^0(\tau), q^0(0))$, whereas the market would take at least one additional period to clear if $\tau = 0$. Intuitively, since the transfer $\tau$ increases the payoff from offering $p_h$, buyers are more willing to offer the high price given any fraction of lemons in the market.

However, the policy under consideration has a second, opposing effect. Since buyers are more willing to offer $p_h$ when they are partially insured against buying a lemon, the average price sellers receive in the future increases as $\tau$ grows larger. Ceteris paribus, this makes sellers more likely to reject offers of $p_\ell$ in early rounds of trade, opting instead to wait for larger payoffs later in the game. To see this, let $\xi^1(q_0, \tau)$ be the mass of buyers who offer $p_h$ in the first period of trade in a 1–step equilibrium when the transfer is $\tau$. The payoff to a type $L$ seller in a 1–step equilibrium is then given by

$$v^1_L(q_0, \tau) = \xi^1(q_0, \tau)p_h + (1 - \xi^1(q_0, \tau)) \int \max\{p_\ell, \delta v^0_L\} dF(\delta).$$

Straightforward algebra shows that $\xi^1(q_0, \tau)$, and thus $v^1_L(q_0, \tau)$, are increasing in $\tau$.

Now observe that $q^2(\tau)$ satisfies

$$q^+ (q^2(\tau), v^1_L(q^1(\tau), \tau)) = q^1(\tau).$$

(23)

Thus, as $\tau$ increases, two opposing forces are at work. On the one hand, since $q^1$ is decreasing in $\tau$, this tends to decrease $q^2$ as well; holding $v^1_L$ constant in (23), $q^2$ is decreasing in $q^1$. On the other hand, holding $q^1$ fixed, $v^1_L$ is increasing in $\tau$, which tends to make sellers more likely to reject an offer of $p_\ell$ at $t = 0$. This implies a smaller increase in the fraction of high quality assets, and hence a larger value of $q^2$. The second effect is not present in a 1–step equilibrium since $v^0_L$ is constant in $\tau$, which explains why $q^1$ is unambiguously decreasing in $\tau$. However, when the market is two or more periods away from market–clearing, which is typically the case when the fraction of high quality assets is small, the second effect is active, and can even dominate the first effect. In other words, when the lemons problem is severe, subsidizing the purchase of assets can increase the time required for market clearing, thus making the market less liquid. These considerations extend to $k$–step equilibria, with $k \geq 3$. 33
Notice that the tension described above does not depend on the assumption that a buyer can offer only one of two prices. Indeed, when buyers can offer any price, subsidizing purchases will still have no effect on current prices for small values of \( q_0 \), as all buyers optimally choose to offer a price strictly less than \( u_L \) (see the discussion in the next section), which type \( H \) sellers reject. However, in later periods, when \( q_t \) is larger, this program can induce more buyers to offer a price that will be accepted by type \( H \) sellers. Thus, as in the case with two prices, future payoffs can increase more than current payoffs, which slows down market recovery.

Returning to our benchmark model and the numerical example from the previous section, Table 1 below summarizes the effect of a transfer \( \tau \) that is equal to 25\% of the loss from purchasing a lemon at price \( p_h \), relative to the benchmark of \( \tau = 0 \).

\[
\begin{array}{cccccccc}
\text{Policy} & q_3 & q_4 & q_3 & q_1 & q_2 & q_0 & q_1 \\
\tau = 0 & 0 & .036 & .206 & .344 & .379 & .410 & .422 & .455 \\
\tau = .25(p_h - u_L) & 0 & .049 & .231 & .301 & .340 & .369 & .382 & .412 \\
\end{array}
\]

Table 1: Policy Analysis

One can see immediately that this policy allows markets to clear faster if the initial fraction of high quality assets is large, but it has little effect on (and can even increase) the time to market clearing if this fraction is small. Consider, for example, an economy with \( q_0 = .4 \): under this policy there exists an equilibrium that clears at \( t = 0 \), whereas it takes until at least \( t = 1 \) for the market to clear in the absence of this policy. However, the opposite is true for, say, \( q_0 = .22 \): the policy increases the minimum number of periods before the market clears from two to three. Thus, even without considering the cost of this type of intervention, we see that its efficacy depends crucially on the underlying severity of the lemons problem. More generally, this exercise highlights that the duration of any intervention is crucial in a dynamic environment with forward–looking agents, and suggests that policymakers might want to consider, for example, using sunset clauses when designing an intervention.\(^{27}\)

\(^{26}\)Given the purpose of this (counter–)example, the fact that it is easily obtained in a simple numerical exercise should help convince the reader that the dynamics discussed in this section are a robust feature of these types of environments.

\(^{27}\)Note that there would be a second effect if we introduced speculators that were more patient than sellers; in this case, these agents might purchase assets early from sellers in order to take advantage of increased
dimension of optimal policymaking has received little attention in the literature, in large part because existing theoretical studies have abstracted from non–stationary dynamics (or restricted attention to one–time interventions).

7 Assumptions

We make several assumptions in our model that allow for a complete analytical characterization of the equilibrium set. In this section, we discuss two important restrictions, and how our results might change if they were relaxed. The first is the assumption that a buyer is restricted to offer one of two (exogenously) fixed prices, and the second is the assumption that all agents exit the market after trading.

Prices

We begin with the assumption of two fixed prices, and argue that it captures the key trade–off that buyers face when deciding on an offer. We then discuss the features of our equilibrium characterization that are preserved when this restriction is relaxed.

Suppose buyers are free to choose any price $p$ when matched with a seller. Since agents discount the future, it is easy to show that all sellers accept any offer $p \geq c_H$, so that no buyer offers more than $c_H$ in equilibrium (we prove this in the Supplementary Appendix). Now note that a buyer never offers $p \in (u_L, c_H)$ in equilibrium, as only type $L$ sellers accept such an offer, and the buyer would receive a negative payoff. Therefore, a buyer effectively chooses between offering $p = c_H$ and a price that maximizes

$$(1 - q)F\left(\frac{p}{v_L}\right)[u_L - p] + \left\{q + (1 - q)\left[1 - F\left(\frac{p}{v_L}\right)\right]\right\} \delta v_B,$$

where, as before, $v_L > 0$ and $v_B > 0$ are the continuation payoffs to type $L$ sellers and buyers, respectively. For ease of exposition, assume $F$ is concave, so that the value of $p$ that maximizes (24) is unique.\(^{28}\) Further abusing notation, let us denote this solution by prices at a later date. This is the “announcement effect” identified in Chiu and Koepppl (2009).

\(^{28}\)When $F$ is not concave, one can show that the solution to (24) is unique almost everywhere (i.e., at all but a countable number of values of $\delta$). We prove this formally in the Supplemental Appendix.
\(p_{\ell}(\delta, v_L, v_B)\); one can show that \(p_{\ell}\) is independent of \(q\) and decreasing in \(\delta\). Given these properties, it follows that there exists a unique value \(\delta^* \in [\delta, \tilde{\delta}]\) such that buyers prefer to offer \(p = c_H\) for \(\delta < \delta^*\) and \(p = p_{\ell}(\delta, v_L, v_B)\) otherwise.

Thus, in the environment with fully flexible prices, when a buyer contemplates an optimal price offer, he faces the same tradeoff captured in the model with two fixed prices: either offer a high price that is accepted by all sellers and trade immediately, or a low price that will be accepted only by sufficiently impatient type \(L\) sellers. The crucial difference here is that the low price is sensitive to the buyer’s discount factor, as well as the continuation payoffs of both buyers and type \(L\) sellers.

Nevertheless, several crucial results from our benchmark model can be established in this more general environment. First, one can show that the fraction of high quality assets increases over time until the market clears. As in the case with exogenous prices, this implies that the market clears in finite time in every equilibrium; see the Supplemental Appendix for formal proofs. Second, one can show that many of the key characteristics of our equilibrium characterization remain true in this more general setting. Using the same argument as in Section 4, one can show that there exists a unique \(q^0 \in (0, 1)\) such that a 0–step equilibrium exists if, and only if, \(q_0 \in [q^0, 1)\). Then, using the revised law of motion for the fraction of high quality assets in the market,

\[
q^+(q, v_L, v_B) = \frac{q[1 - F(\delta^*)]}{q[1 - F(\delta^*)] + (1 - q) \left[ 1 - F(\delta^*) - \int_{\delta^*}^{\tilde{\delta}} F(p_{\ell}(\delta, v_L, v_B)/v_L) dF(\delta) \right]},
\]

one can follow the recursive procedure in Section 4 to derive the necessary and sufficient conditions for a \(k\)-step equilibrium for \(k \geq 1\):

\[
q^+ \left( q_0, v_{L}^{k-1}(q'), v_{B}^{k-1}(q') \right) = q'; \quad (25)
\]

\[
q' \in [q^{k-1}, v_{L}^{k-1}] \cap (0, 1); \quad (26)
\]

\[
\pi^B_B(q) < \pi^B_{\ell} \left( q_0, \tilde{\delta}, v_{L}^{k-1}(q'), v_{B}^{k-1}(q') \right). \quad (27)
\]

\(^{29}\)Since \(v_L \leq c_H\) and \(v_B \leq u_H - c_H\), a sufficient condition for \(p_{\ell}(\delta, v_L, v_B) > \delta v_L\) is \(u_L > \delta c_H + \tilde{\delta}(u_H - c_H)\). This inequality is the analog of (5) in the model with two prices. Thus, even in an environment with fully flexible prices, there exists a region of the parameter space in which buyers never make an offer that is rejected with probability one.
Conditions (25) to (27) are analogous to conditions (16) to (18) in our benchmark model with two fixed prices. Crucially, the fixed point mapping described in (16) was shown to be single-valued, continuous, and strictly increasing in $q_0$. Establishing these properties for the analogous mapping described in (25) would allow for the same clean characterization of equilibria in this more general setting. Unfortunately, though these properties appear to be satisfied in a variety of numerical simulations, we cannot verify them analytically; since the law of motion depends explicitly on buyers’ offers, and these offers in turn depend on future payoffs (which depend on both $q'$ and future offers), the analysis quickly becomes considerably more complex. However, one can analytically derive the set of 0-step and 1-step equilibrium, and show formally that these sets have all of the same properties as they do in the environment described in the main text. In particular, $q_1 < q^0 < q^1$, which confirms that the existence of multiple equilibria derives from deep features of the economic environment, and not from the restriction to two prices.  

**One-Time Entry**

In order to study how markets clear on their own in the simplest possible environment, we assume there is a fixed stock of buyers and sellers, and that these agents leave the market after trading. In doing so, we abstract from several interesting issues. For example, one might want to allow buyers to re-sell their asset, either because they discover it is of low quality or because of some stochastic, exogenous taste shock (as in Chiu and Koepppl (2009)). The effect of allowing for re-sale depends crucially on the information structure.

Suppose, for instance, that agents observe the history of trade for a particular asset.  

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30 In the model with fully flexible prices, one can also analyze whether equilibrium outcomes become efficient as trading frictions vanish, i.e., as the time interval between two consecutive trading opportunities converges to zero. It is possible to show that the answer is “no”: the amount of time it takes for the market to clear does not converge to zero as trading frictions vanish. This result is not surprising. We know that when adverse selection is present, real inefficiencies can persist as trading frictions vanish; see Janssen and Roy (2002), Deneckere and Liang (2006), and Hörner and Vieille (2009).

31 This is possible in many markets, either because there are relatively few agents in the market (as in some markets for very specific financial assets), or because there exists a technology that keeps track of such histories (such as Carfax, in the used car market). However, the exact nature of the information available is crucial; for example, Kim (2011) illustrates the difference between observing the number of previous meetings and observing the time on the market.
Consider first the case in which there are no taste shocks, and so buyers who purchase a high quality asset have no reason to re-sell it. In this case, agents can infer that an asset is being re-sold precisely because it is low quality, leaving no expected gains from trade. Thus, buyers who purchase a lemon have no incentive to attempt to re-sell it even if this is feasible. Consider now the case in which there are taste shocks, so that the decision to re-sell an asset does not necessarily signal that it is low quality. While the analysis in this case would certainly be more complicated, we conjecture that allowing re-sale makes markets less liquid than they are in the baseline model. Intuitively, since buyers may need to re-sell a high quality asset that they purchase, they should be less willing to offer $p_h$ to begin with (because of adverse selection in the re-sale market), thus decreasing the liquidity of these assets.

In addition to allowing buyers to re-enter the market, one might also want to consider what would happen if new sellers could generate assets at some cost and enter the market over time. Again, the consequences of this extension depend heavily on assumptions about the properties of an asset that are observable. For instance, if an asset’s date of creation or “vintage” is observable, one could imagine a constant inflow of new vintages at every date, where the trading dynamics of each vintage resembles those of the single vintage we consider in our baseline model. Alternatively, if vintages are not observable, an entry condition could be used to endogenously determine the composition of high and low quality assets in the market. In fact, we think the question of asset generation is extremely important—particularly in the context of policy analysis—and our model is ideally situated to address this issue. This is the focus of current work.

8 Conclusion

This paper provides a theory of how markets suffering from adverse selection can recover over time on their own. Sellers with low quality assets exit the market relatively more quickly than those with high quality assets, causing the average quality of assets in the market to increase over time. Eventually, all assets are exchanged. The model delivers sharp
predictions about how long this process takes, or the extent to which the market is illiquid, as well as the behavior of prices over time. Interestingly, we find multiple equilibria, which suggests that there is scope for coordination failures in dynamic, decentralized markets with adverse selection. We argue that this model serves as a useful benchmark for understanding how exogenous events or interventions will affect the speed with which markets recover. We provide a specific example from the recent financial crisis, and show how accounting for dynamic considerations can shed light on potentially harmful, unintended effects of policies aimed at restoring liquidity in frozen markets. Natural extensions include allowing sellers the choice of what type of asset to generate and when to enter the market, allowing buyers to acquire costly information about an asset’s quality, and introducing aggregate uncertainty and learning. These are left for future work.
Appendix: Omitted Proofs and Intermediate Results

Constructing Payoffs (Section 2)

Here we show how to derive payoffs using our refinement. We make use of the following result, the (straightforward) proof of which we omit: Suppose \( f : (0, 1) \to \mathbb{R} \) is bounded and uniformly continuous. Then, the limit \( \lim_{\alpha \to 1} f(x) \) exists.

Fix a strategy profile \( \sigma \) and let \( a = \{a_t^j\} \) be a strategy for a type \( j \) seller. The payoffs \( V^j_t(a|\sigma, \alpha) \) are well–defined for all \( \alpha \in (0, 1) \) and satisfy the following recursion:

\[
V^j_t(a|\sigma, \alpha) = (1 - \alpha) \int \delta V^j_{t+1}(a|\sigma, \alpha) dF(\delta) + \alpha \sum_{p \in \{p_L, p_H\}} \xi_t(p|\sigma) \int \{a_t^j(\delta, p)(p - c_j) + [1 - a_t^j(\delta, p)] \delta V^j_{t+1}(a|\sigma, \alpha)\} dF(\delta),
\]

where \( \xi_t(p|\sigma) \) is the fraction of buyers who offer \( p \) in period \( t \). In words, after drawing his discount factor, the seller gets an opportunity to trade with probability \( \alpha \), in which case he receives an offer \( p \in \{p_L, p_H\} \) with probability \( \xi_t(p|\sigma) \). If he accepts the offer, i.e., if \( a_t^j(\delta, p) = 1 \), then he obtains a payoff of \( p - c_j \). If he rejects the offer or if he does not get an opportunity to trade, then he stays in the market and obtains a payoff of \( \delta V^j_{t+1}(a|\sigma, \alpha) \). A standard argument shows that the payoffs \( V^j_t(a|\sigma, \alpha) \) are bounded and uniformly continuous in \( \alpha \), so that the limits \( \lim_{\alpha \to 1} V^j_t(a|\sigma, \alpha)\) exist.

For any strategy \( p = \{p_t\} \) for a buyer, the payoffs \( V^B_t(p|\sigma, \alpha) \) are also well–defined for all \( \alpha \in (0, 1) \) and satisfy the following recursion:

\[
V^B_t(p|\sigma, \alpha) = (1 - \alpha) \int \delta V^B_{t+1}(p|\sigma, \alpha) dF(\delta) + \alpha \int \{q_t(\sigma, \alpha)A^H_t(p_t(\delta)|\sigma)[u_H - p_t(\delta)] + (1 - q_t(\sigma, \alpha))A^L_t(p_t(\delta))[u_L - p_t(\delta)] + [1 - q_tA^H_t(p_t(\delta)) - (1 - q_t)A^L_t(p_t(\delta))]\delta V^B_{t+1}(p|\sigma, \alpha)\} dF(\delta),
\]

where \( q_t(\sigma, \alpha) \) is the fraction of type \( H \) sellers in the market in period \( t \) as a function of \( \alpha \) (and the strategy profile \( \sigma \)). In words, after drawing his discount factor, the buyer gets an opportunity to trade with probability \( \alpha \), in which case he offers \( p_t(\delta) \). If he is matched with a type \( j \) seller, his offer is accepted with probability \( A^j_t(p_t(\delta)) \), in which case he obtains a
payoff of \( u_j - p_t(\delta) \). If his offer is rejected, or if he does not get an opportunity to trade, then he stays in the market and obtains a payoff of \( \delta V_{t+1}^B(p) \). As above, a standard argument shows that the payoffs \( V_t^B(p) \) are bounded and uniformly continuous in \( \alpha \), so that the limits \( \lim_{\alpha \to 1} V_t^B(p) \) exist as well.

**Proof of Lemma 3**

Let \( \sigma^* \) be an equilibrium and assume, toward a contradiction, that \( T(\sigma^*) = \infty \). First notice that there exists \( q^* \in (0, 1) \) such that

\[
q^*[u_H - p_h] + (1 - q^*)[u_L - p_h] = u_L - p_t. \tag{28}
\]

Since \( \delta[u_H - p_h] \leq u_L - p_t \), the right side of (28) is an upper bound for the payoff a buyer can obtain if he offers \( p_t \). Hence, if the fraction of type \( H \) sellers in the market is above \( q^* \), then all buyers offer \( p_h \) and the market clears. Consequently, for all \( t \geq 0 \), \( q_t \) is bounded above by \( q^* \), and so is the limit of the sequence \( \{q_t\}_{t=0}^\infty \). Now observe that since \( V_t^L(\sigma^*) \leq p_h \), we have that \( F(p_t/V_t^L(\sigma^*)) \geq F(p_t/p_h) \) for all \( t \geq 1 \). Thus, the law of motion (9) implies that

\[
q_t \geq \frac{q_0}{q_0 + (1 - q_0)[1 - F(p_t/p_h)]^t}.
\]

However, the right side of the above equation converges to one, a contradiction. \( \blacksquare \)

**Proof of Proposition 1**

Let \( \eta^0(q_0, \delta) = \pi_h^B(q_0) - \pi_t^B(q, \delta, v_L^0, v_B^0(q_0)) \). Note that

\[
\eta^0(q_0, \delta) = (1 - \delta)v_B^0(q_0) - (1 - q_0)F\left(\frac{p_t}{p_h}\right)[u_L - p_t - \delta v_B^0(q_0)].
\]

Since \( v_B^0(q) \leq u_H - p_h \) and \( u_L - p_t \geq \delta[u_H - p_h] \), we then have that

\[
\frac{\partial \eta^0}{\partial q_0}(q, \delta) = (1 - \delta)[u_H - u_L] + F\left(\frac{p_t}{p_h}\right)[u_L - p_t - \delta v_B^0(q)] + (1 - q)F\left(\frac{p_t}{p_h}\right)\delta[u_H - u_L] > 0.
\]

Now observe that \( \eta^0(0) < 0 < \eta^0(1) \) and \( \eta^0 \) is continuous. So, there exists a unique \( q_0 \in (0, 1) \) such that \( \eta^0(q_0, \delta) \geq 0 \) if, and only if, \( q_0 \geq q_0^0 \). Moreover, \( \eta(q_0^0, \delta) = 0 \) implies that

\[
v_B^0(q_0^0)\left[1 - \delta + (1 - q_0^0)F\left(\frac{p_t}{p_h}\right)\delta\right] = (1 - q_0^0)F\left(\frac{p_t}{p_h}\right)[u_L - p_t],
\]
and so $v_B(q^0) > 0$. Thus, $σ^0$ is an equilibrium if, and only if, $q_0 ∈ [q^0, 1)$.

Suppose now $q_0 < q^0$ and consider a strategy profile $σ^0$ with the necessary property that all buyers offer $p_h$ in $t = 0$. One alternative for a buyer is to offer $p_h$ in every period regardless of his discount factor. Let $\tilde{p}$ denote this strategy. It must be that $V_t^B(σ^0) ≥ V_t^B(\tilde{p}|σ^0)$ for all $t ≥ 0$ if $σ^0$ is to be an equilibrium. Now observe that if the probability of trade in each period is $α ∈ (0, 1)$, then

$$V_t^B(\tilde{p}|σ^0, α) = \sum_{τ=1}^{∞} α(1− α)^{τ−1}E[δ]^{τ−1}v_B(q^{α}_{t+τ−1})$$

for all $t ≥ 0$, where $q^{α}_{t+τ−1}$ is the fraction of type $H$ sellers in the market in period $t + τ − 1$. It is easy to see that the sequence $\{q^{α}_{t}\}_{t=0}^{∞}$ is non-decreasing. Hence,

$$V_t^B(\tilde{p}|σ^0, α) ≥ \sum_{τ=1}^{∞} α(1− α)^{τ−1}E[δ]^{τ−1}v_B(q_0),$$

which implies that $V_t^B(\tilde{p}|σ^0) ≥ v_B(q_0)$. Thus, $σ^0$ is an equilibrium only if $V_t^B(σ^0) ≥ v_B(q_0)$. However, since $V_t^L(σ^0) ≤ p_h$ and $q_0 < q^0$ implies that $η_0(q_0) < 0$, we have that

$$π_L^B(q_0, δ, V_1^L(σ^0), V_1^B(σ^0)) ≥ π_L^B(q_0, δ, v_L^0, v_B(q_0)) > π_h^B(q_0)$$

for all $q_0 < q^0$. Therefore, there exists $δ' < δ$ such that it can not be optimal for a buyer with discount factor in $(δ', δ]$ to offer $p_h$ at $t = 0$, so that the market clearing immediately cannot be an equilibrium outcome.

\textbf{Proof of Proposition 2}

Recall that $q^+(q, v_L^0)$ is strictly increasing in $q$ when $p_L/v_L^0 < δ$ and that $q^+(q, v_L^0) ≡ 1$ otherwise. Therefore, there exists a unique $q^1 < q^0$ such that $q^+(q_0, v_L^0) ≥ q^0$ if, and only if, $q_0 ∈ [q^1, 1]$. Note that $q^1 = 0$ if $p_L/v_L^0 ≥ δ$ and $q^1$ is such that $q^+(q^1, v_L^0) = q^0$ otherwise. Now let $η_1(q_0, δ) = π_h^B(q_0) - π_L^B(q_0, δ, v_L^0, v_B(q^+(q_0, v_L^0)))$. Straightforward algebra shows that

$$\frac{∂η_1}{∂q_0}(q, δ) = F \left( \frac{p_L}{p_h} \right) \left\{ (u_L - p_L - δv_B(q^+(q, p_h))) \right\}$$

$$+(u_H - u_L) \left\{ 1 - δ \left\{ q + (1 - q) \left[ 1 - F \left( \frac{p_L}{p_h} \right) \right] \right\} \frac{∂q^+}{∂q}(q, p_h) \right\}.$$
Thus, since \( u_L - p_L \geq \delta |u_H - p_H| \) and
\[
\left\{ q + (1 - q) \left[ 1 - F \left( \frac{p_L}{p_H} \right) \right] \right\} \frac{\partial q^+}{\partial q}(q, p_H) = 1 - \frac{qF (p_L/p_H)}{q + (1 - q) [1 - F (p_L/p_H)]} < 1,
\]
we can then conclude that \( \eta^1 \) is strictly increasing in \( q_0 \) regardless of the value of \( p_L/p_H \). Since
\( \eta^1(0, \delta) < 0 < \eta^1(1, \delta) \) and \( \eta^1 \) is continuous in \( q_0 \), there exists a unique \( q^1 \in (0, 1) \) such that
\( \eta^1(q_0, \delta) < 0 \) if, and only if \( q_0 \in [0, q^1) \). Hence, \( \pi^B_h(q_0) < \pi^B_L(q_0, \delta, v^0_L, v^0_B(q^0, q^0, v^0_B(q^0, q^0))) \) if, and only if \( q_0 \in [0, q^1) \). Next, observe that since \( v^0_B(q^0, p_H) > v^0_B(q_0) \) for all \( q_0 \in (0, 1) \),
\[
\pi^B_L(q_0, \delta, v^0_L, v^0_B(q^0, p_H)) > \pi^B_L(q_0, \delta, v^0_L, v^0_B(q^0)) = \pi^B(q_0^0).
\]
Thus, \( \eta^1(q_0^0, \delta) < 0 \), from which we obtain that \( q^1 > q_0^0 \).

\[\square\]

**Lemma 4 and Proof**

**Lemma 4.** The payoff \( v^1_B \) is continuous in \( q_0 \) and \( v^1_B(q_0) - v^1_B(q_0) \leq (q_0' - q_0)[u_H - u_L] \) for all \( q_0' > q_0 \). The fraction \( \xi^1 \) is continuous and increasing in \( q_0 \), with \( \lim_{q_0 \to q^1} \xi^1(q_0) = 1 \). The payoff \( v^1_L \) is continuous and increasing in \( q_0 \), with \( \lim_{q_0 \to q^1} v^1_L(q_0) = v^0_L \).

**Proof:** Note that \( q^+(q_0, v^0_L) \) continuous in \( q_0 \) implies that \( v^1_B(q_0) \) is also continuous in \( q_0 \).

We now prove that \( v^1_B(q_0') - v^1_B(q_0) \leq (q_0' - q_0)[u_H - u_L] \) for all \( q_0' > q_0 \). First, note that
\[
\pi^B_L(q, \delta, v^0_L, \pi^B_h[q^+(q, v^0_L)]) = \delta \pi^B_L(q) + (1 - q) F \left( \frac{p_L}{v_L} \right) [u_L - p_L - \delta (u_L - p_H)]
\]
for all \( q \in (0, 1) \) and \( \delta \in [\delta, \bar{\delta}] \). This fact is useful in what follows. Now let
\[
v^1_B(q_0, \delta) = \pi^B_L(q_0, \delta, v^0_L, v^0_B(Q^1_B(q_0))) + \max\{\eta^1(q_0, \delta), 0\},
\]
where \( \eta^1(q_0, \delta) \) is defined in the proof of Proposition 2. Since \( v^1_B(q_0) = \int v^1_B(q_0, \delta) dF(\delta) \), we are done if we show that \( q_0' > q_0 \) implies that
\[
v^1_B(q_0', \delta) - v^1_B(q_0, \delta) \leq (q_0' - q_0)[u_H - u_L]
\]
regardless of \( \delta \). We know from the proof of Proposition 2 that \( \eta^1 \) is strictly increasing in \( q_0 \). Since \( \eta^1(0, \delta) < 0 < \eta^1(1, \delta) \) for all \( \delta \in [\delta, \bar{\delta}] \), we have that for each \( \delta \in [\delta, \bar{\delta}] \) there exists a
unique \( q^* = q^*(\delta) \in (0, 1) \) such that \( \eta^1(q^*, \delta) \geq 0 \) if, and only if, \( q \geq q^* \); note that \( q^*(\overline{\delta}) = \overline{q}^1 \).

Now let \( q'_0 > q_0 \). Since

\[
v_B^1(q'_0, \delta) - v_B^1(q_0, \delta) \leq v_B^1(q'_0, \delta) - \pi^B_h(q_0),
\]

we have that (30) holds if \( q'_0 > q^* \). Suppose then that \( q'_0 \leq q^* \). In this case, by (29),

\[
v_B^1(q'_0, \delta) - v_B^1(q_0, \delta) = \delta[\pi^B_h(q'_0) - \pi^B_h(q_0)] + (q_0 - q'_0) F\left(\frac{p}{v_L^0}\right) [u_L - p - \delta(u_L - p_h)],
\]

from which the desired result follows given that the second term on the right side of the above equation is negative.

Next, we prove that \( \xi^1(q_0) \) is continuous and increasing in \( q_0 \), with \( \lim_{q_0 \to \overline{q}^1} \xi^1(q_0) = 1 \). Given that \( \pi^B_F(q_0, \delta, v_L^0, v_B^0(Q_1^1(q_0))) \) is strictly increasing in \( \delta \), \( \eta^1 \) is strictly decreasing in \( \delta \). Let \( \delta^1(q_0) \), with \( q_0 \in [q_1, \overline{q}^1) \cap (0, \overline{q}^1) \), be such that: (i) \( \delta^1(q_0) = 0 \) if \( \eta^1(q_0, \delta^1(q_0)) = 0 \); and (ii) \( \eta^1(q_0, \delta^1(q_0)) = 0 \) if \( \eta^1(q_0, 0) > 0 \). Since \( \eta^1(\overline{q}^1, \overline{\delta}) = 0 \) and \( \eta^1 \) is strictly increasing in \( q \), \( \delta^1(q_0) \) is uniquely defined. By construction, \( \delta^1 \) is the cutoff discount factor below which a buyer finds it strictly optimal to offer \( p_h \) in \( t = 0 \). Hence, the probability \( \xi^1(q_0) \) that a buyer offers \( p_h \) in \( t = 0 \) is equal to \( F(\delta^1(q_0)) \). Given that \( \eta^1 \) is jointly continuous, a standard argument shows that \( \delta^1 \) depends continuously on \( q_0 \). Moreover, the cutoff \( \delta^1(q_0) \) is strictly increasing in \( q_0 \) if \( \eta^1(q_0, 0) > 0 \), as \( \eta^1 \) is strictly increasing in \( q \). The desired result follows from the fact that \( F \) is continuous and strictly increasing and \( \lim_{q_0 \to \overline{q}^1} \delta^1(q_0) = \overline{\delta} \) (as \( \eta^1(\overline{q}^1, \overline{\delta}) = 0 \)).

To finish the proof, note that the continuity of \( \xi^1(q_0) \) and the fact that \( \lim_{q_0 \to \overline{q}^1} \xi^1(q_0) = 1 \) imply that \( v_L^1(q_0) \) is continuous in \( q_0 \), with \( \lim_{q_0 \to \overline{q}^1} v_L^0(q_0) = p_h = v_L^0. \)

\textbf{Proof of Proposition 3}

We first show that (16) and (17) imply (18), so that the first two conditions completely determine the range of values of \( q_0 \) for which there exists a 2-step equilibrium. Suppose that \( q' \in [\underline{q}^1, \overline{q}^1) \cap (0, 1) \). In order to prove that (18) is satisfied, it is sufficient to show

\[
\pi^B_h(q') - \pi^B_h(q_0) \geq \pi^B_F(q', \delta, v_L^0, v_B^0(q^+(q'), v_L^0(q'_0))) - \pi^B_h(q_0, \delta, v_L^1(q'), v_B^1(q'))
\]

for all \( \delta \in [\delta, \overline{\delta}] \). Condition (31) implies that, no matter his discount factor, the incentive of a buyer to choose \( p_e \) in \( t = 0 \) is even greater than his incentive to choose \( p_e \) in \( t = 1 \), when
the fraction of type $H$ sellers in the market is $q' > q_0$. First, note from (29) in the proof of Lemma 4 that

$$
\pi^B_\ell (q', \tilde{\delta}, v_L^0, v_B^0(q^+(q', v_L^0))) = \delta \pi^B_h(q') + (1 - q') F(\frac{p_\ell}{v_L^0}) \left[ u_L - p_\ell - \tilde{\delta}(u_L - p_h) \right].
$$

Second, since $v_B^0(q') \geq \pi^B_h(q')$, we have

$$
\pi^B_\ell (q_0, \tilde{\delta}, v_L^1(q'), v_B^1(q')) \geq \pi^B_\ell (q_0, \tilde{\delta}, v_L^1(q'), \pi^B_h(q'))
= \delta \pi^B_h(q_0) + (1 - q_0) F\left(\frac{p_\ell}{v_L^1(q')}\right) \left[ u_L - p_\ell - \tilde{\delta}(u_L - p_h) \right];
$$

the second equality follows from (29) and the fact that $q' = q^+(q_0, v_L^1(q'))$ by (16). Therefore,

$$
\pi^B_\ell (q', \tilde{\delta}, v_L^0, v_B^0(q^+(q', v_L^0))) - \pi^B_\ell (q_0, \tilde{\delta}, v_L^1(q'), v_B^1(q'))
\leq \tilde{\delta} [\pi^B_h(q') - \pi^B_h(q_0)] + \left\{ (1 - q') F\left(\frac{p_\ell}{v_L^0}\right) - (1 - q_0) F\left(\frac{p_\ell}{v_L^1(q')}\right) \right\} \left[ u_L - p_\ell - \tilde{\delta}(u_L - p_h) \right] .
$$

Since $v_L^0 > v_L^1(q')$ for all $q' \in [\underline{q}, \overline{q}] \cap (0, 1)$, $u_L < p_h$, and $q' > q_0$, the second term on the right side of the above inequality is negative, which confirms (31).

We now show that there exists a 2-step equilibrium if, and only if, $q_0 \in [\underline{q}, \overline{q}] \cap (0, 1)$. First note that since $\overline{q} < 1$, (16) and (17) can be satisfied only if the denominator of

$$
q^+(q_0, v_L^1(q')) = \frac{q_0}{q_0 + (1 - q_0)[1 - F(p_\ell/v_L^1(q'))]}
$$

is greater than $q_0$, i.e., only if $p_\ell/v_L^1(q') < \tilde{\delta}$. Now observe that if $p_\ell/v_L^1(q') < \tilde{\delta}$, then

$$
q^-(q') = \frac{q' [1 - F(p_\ell/v_L^1(q'))]}{1 - q' F(p_\ell/v_L^1(q'))}
$$

belongs to the interval $(0, 1)$ and is such that $q^+(q^-(q'), v_L^1(q')) = q'$. Thus, (16) is satisfied for $q' \in [\underline{q}, \overline{q}] \cap (0, 1)$ if, and only if, $p_\ell/v_L^1(q') < \tilde{\delta}$. Moreover, it is immediate to see that $q^-(q')$ is the only possible value of $q_0$ for which (16) and (17) can hold.

Since $v_L^1(q')$ is increasing in $q'$, $p_\ell/v_L^1(\overline{q}) < \tilde{\delta}$ implies that $p_\ell/v_L^1(q') < \tilde{\delta}$ for all $q' > \overline{q}$. Let then $\overline{q}$ be such that $\overline{q} = 0$ if $p_\ell/v_L^1(\overline{q}) < \tilde{\delta}$ and $\overline{q} = \sup\{q' \in [\underline{q}, \overline{q}] : p_\ell/v_L^1(q') = \tilde{\delta}\}$ if $p_\ell/v_L^1(\overline{q}) \geq \tilde{\delta}$; note that $\overline{q}$ is well-defined since $p_\ell/v_L^1(\overline{q}) = p_\ell/v_L^0 < \tilde{\delta}$. By construction, there exists $q_0 \in (0, 1)$ such that (16) and (17) are satisfied if, and only if, $q' \in [\underline{q}, \overline{q}] \cap (\overline{q}, 1)$, in which case $q_0 = q^-(q')$. Given that $F$ and $v_L^1$ are continuous, it is easy to see that $q^-$ is
continuous. Moreover, since $v_L$ is increasing in $q'$, the map $q^-$ is also strictly increasing in $q'$. Thus, $q^-$ is invertible and its inverse $Q^2_+ : [q^1, q^1] \cap (q^1, 1) \to (0, 1)$ is continuous and strictly increasing. By construction, we have that: (i) when $p_e/v_L(q^1) < \delta$, a 2–step equilibrium exists if, and only if, $Q^2_+(q_0) \in [q^1, q^1] \cap (0, 1)$; (ii) when $p_e/v_L(q^1) \geq \delta$, a 2–step equilibrium exists if, and only if, $Q^2_+(q_0) \in (q^1, q^1)$. We are done if we show that $\lim_{q' \to q^1} q^-(q') = 0$ when $p_e/v_L(q^1) \geq \delta$. This follows from the fact that $\lim_{q' \to q^1} F(p_e/v_L(q')) = 1$. 

**Lemma 5 and Proof**

**Lemma 5.** The payoff $v_B^2$ is continuous in $q_0$ and $v_B^2(q_0') - v_B^2(q_0) \leq (q_0' - q_0)[u_H - u_L]$ for all $q_0' > q_0$. The fraction $\xi^2$ is continuous and increasing in $q_0$, with $\lim_{q_0 \to -q^2} \xi^2(q_0) = \xi^1(q^2)$. The payoff $v_L^2$ is continuous and increasing in $q_0$, with $v_L^2(q^2) \equiv \lim_{q_0 \to -q^2} v_L^2(q_0) = v_L^1(q^2)$ and $v_L^2(q_0) \leq v_L^1(Q^2_+(q_0))$ for all $q_0$.

**Proof:** We start by showing that $\eta^2(q_0, \delta) = \pi^B_\ell(q_0) - \pi^B_\ell(q_0, \delta, v_L^1(Q^2_+(q_0)), v_B^1(Q^2_+(q_0)))$ is strictly increasing in $q_0$; this is important in what follows. Let $q'_0 > q_0$ and note that

$$
\pi^B_\ell(q'_0, \delta, v_L^1(Q^2_+(q'_0)), v_B^1(Q^2_+(q'_0))) - \pi^B_\ell(q_0, \delta, v_L^1(Q^2_+(q_0)), v_B^1(Q^2_+(q_0)))
= \left\{ q'_0 + (1 - q'_0) \left[ 1 - F \left( \frac{p_e}{v_L^1(Q^2_+(q'_0))} \right) \right] \right\} \delta \left[ v_B^1(Q^2_+(q'_0)) - v_B^1(Q^2_+(q_0)) \right]
+ \left[ \delta v_B^1(Q^2_+(q_0)) - (u_L - p_e) \right] \left\{ (1 - q_0)F \left( \frac{p_e}{v_L^1(Q^2_+(q_0))} \right) - (1 - q'_0)F \left( \frac{p_e}{v_L^1(Q^2_+(q'_0))} \right) \right\}
\leq \left\{ q'_0 + (1 - q'_0) \left[ 1 - F \left( \frac{p_e}{v_L^1(Q^2_+(q'_0))} \right) \right] \right\} \delta \left[ Q^2_+(q_0') - Q^2_+(q_0) \right] (u_H - u_L);
$$

the inequality follows from the fact that $v_B^1(q'_0) - v_B^1(q_0) \leq (q'_0 - q_0)[u_H - u_L]$ for all $q'_0 > q_0$ and the fact that $Q^2_+(q_0)$ is increasing in $q_0$. Now observe that $Q^2_+(q_0)$ and $v_L^1(q_0)$ increasing in $q_0$ together imply that

$$
q_0 + (1 - q_0) \left[ 1 - F \left( \frac{p_e}{v_L^1(Q^2_+(q_0))} \right) \right] = 1 - (1 - q_0)F \left( \frac{p_e}{v_L^1(Q^2_+(q_0))} \right)
$$

is increasing in $q_0$. Hence,

$$
\pi^B_\ell(q'_0, \delta, v_L^1(Q^2_+(q'_0)), v_B^1(Q^2_+(q'_0))) - \pi^B_\ell(q_0, \delta, v_L^1(Q^2_+(q_0)), v_B^1(Q^2_+(q_0))) \leq \delta(q_0' - q_0)[u_H - u_L],
$$

(32)
from which we obtain that \( \eta^2(q_0', \delta) - \eta^2(q_0, \delta) \geq (1 - \delta)(q_0' - q_0)(u_H - u_L) > 0 \). This proves the desired result.

We first establish the properties of \( v^1_B \). Since \( Q^2_+ \) is continuous in \( q_0 \), the continuity of \( v^2_B \) follows from the continuity of \( v^1_B \) and \( v^1_L \). We now prove that if \( q_0' > q_0 \), then \( v^2_B(q_0') - v^2_B(q_0) \leq (q_0' - q_0)[u_H - u_L] \). For this, let

\[
v^2_B(q_0, \delta) = \pi^B(h, q_0, \delta, v^1_L(Q^2_+(q_0))\right) + \max\{\eta^2(q_0, \delta), 0\},
\]

As in the proof of Lemma 4, the result from the previous paragraph implies that for each \( \delta \in [\delta, \overline{\delta}] \), there exists a unique \( q^* = q^*(\delta) \in (0, 1) \) such that \( \eta^2(q_0, \delta) \geq 0 \) if, and only if, \( q_0 \geq q^* \); by construction, \( q^*(\overline{\delta}) = \overline{q}^2 \). Let then \( q_0' > q_0 \). The same argument as in the proof of Lemma 4 shows that

\[
v^2_B(q_0, \delta) - v^2_B(q_0, \delta) \leq (q_0' - q_0)[u_H - u_L]
\]

if \( q_0' > q^* \). Suppose then that \( q_0' \leq q^* \). In this case, the above inequality follows from (32), and the desired result holds from the fact that \( v^2_B(q_0) = \int v^2_B(q_0, \delta) dF(\delta) \).

Now, we establish the properties of \( \xi^2 \). Since \( \eta^2 \) is strictly increasing in \( q_0 \) and strictly decreasing in \( \delta \), an argument similar to the one used in the proof of Proposition 2 shows that for each \( q_0 \in [\overline{q}^2, \overline{q}^2] \cap (0, 1) \), there exists a unique \( \delta^2 = \delta^2(q_0) \in [\delta, \overline{\delta}] \), which is continuous and increasing in \( q_0 \), such that \( \eta^2(q_0, \delta) \geq 0 \) if, and only if \( \delta \leq \delta^2(q_0) \). Thus, \( \xi^2(q_0) = F(\delta^2(q_0)) \) is continuous and increasing in \( q_0 \). Notice that \( \lim_{q_0 \to \overline{q}^2} \xi^2(\overline{q}^2) = \xi^1(\overline{q}^2) \), as

\[
\lim_{q_0 \to \overline{q}^2} \eta^2(q_0, \delta) = \pi^B(\overline{q}^2) - \pi^B(\overline{q}^2, \delta, v^1_B(\overline{q}^2), v^1_B(\overline{q}^1)) = \pi^B(\overline{q}^2) - \pi^B(\overline{q}^2, \delta, v^0_B, v^0_B[\overline{q}^2, v^0_L]) = \eta^1(\overline{q}^2, \delta).
\]

For the properties of \( v^2_L \), first notice that \( Q^2_+ \) and \( v^1_L \) continuous in \( q_0 \) imply that \( v^2_L \) is also continuous in \( q_0 \). Moreover,

\[
\lim_{q_0 \to \overline{q}^2} v^2_L(q_0) = \xi^1(\overline{q}^2)p_h + (1 - \xi^1(\overline{q}^2)) \int \max\{p, \delta v^0_L\} dF(\delta) = v^1_L(\overline{q}^1).
\]

To finish, note that \( v^1_L(q) \leq v^0_L(q, v^0_L) = v^0_L \) implies that

\[
v^2_L(q_0) \leq \xi^2(q_0)p_h + (1 - \xi^2(q_0)) \int \max\{p, \delta v^0_L\} dF(\delta).
\]
Moreover, by (31) in the proof of Lemma 4, we have that \( \xi^2(q_0) \leq \xi^1(Q_+^2(q_0)) \). From this, it is immediate to see that \( v_{\ell}^2(q_0) \leq v_{\ell}^1(Q_+^2(q_0)) \) for all \( q_0 \).

\[ \blacksquare \]

**Proof of Theorem 1**

We proceed by induction. Suppose there exists \( k \geq 3 \) and sequences of cutoffs \( \{q^s\}_{s=0}^{k-1} \) and \( \{\overline{q}^s\}_{s=0}^{k-1} \) such that:

(A1) \( \overline{q}^0 = 1 \) and \( q^s \leq q^{s-1} < \overline{q}^s \) for all \( s \in \{1, \ldots, k-1\} \);

(A2) a \( s \)-step equilibrium, with \( s \in \{0, \ldots, k-1\} \), exists if, and only if, \( q_0 \in [q^s, \overline{q}^{s-1}) \cap (0, 1) \).

Moreover, suppose that for each \( s \in \{0, \ldots, k-1\} \), there exist functions \( v_{B}^s(q_0) \) and \( v_{L}^s(q_0) \), and a map \( Q_{\pm}^s(q_0) \), such that:

(A3) \( Q_{\pm}^s(q_0) \) is the value of \( q_1 \) in any \( s \)-step equilibrium when the initial fraction of type H sellers is \( q_0 \);

(A4) given \( q_0 \in [q^s, \overline{q}^s) \cap (0, 1) \), the payoffs to buyers and type L sellers in a \( s \)-step equilibrium are \( v_{B}^s(q_0) \) and \( v_{L}^s(q_0) \), respectively;

(A5) for all \( s \in \{2, \ldots, k-1\} \), if \( q' = Q_{\pm}^s(q_0) \), then

\[
\eta^s(q', \delta) \leq \eta^s(q_0, \delta) \quad \text{for all} \quad q_0 \in [q^s, \overline{q}^s) \quad \text{and} \quad \delta \in [\delta, \overline{\delta}];
\]

(A6) \( v_{B}^s \) is continuous in \( q_0 \) and such that \( v_{B}^s(q'_0) - v_{B}^s(q_0) \leq (q'_0 - q_0)[u_H - u_L] \) for all \( q'_0 > q_0 \);

(A7) \( v_{B}^s \) is continuous and increasing in \( q_0 \), with \( v_{B}^s(\overline{q}^s) \equiv \lim_{q_0 \to \overline{q}^s} v_{B}^s(q_0) = v_{B}^{s-1}(\overline{q}^s) \) and \( v_{B}^s(q_0) \leq v_{B}^{s-1}(Q_{\pm}^s(q_0)) \) for all \( q_0 \).

Finally, suppose that:

(A8) for each \( s \in \{1, \ldots, k-1\} \), \( q^s = 0 \) if, and only if, \( p_{\ell}/v_{L}^{s-1}(q^{s-1}) \geq \overline{\delta} \).
Conditions (A1) to (A8) are true when $k = 3$ by Propositions 1 to 3 and Lemmas 4 and 5; condition (A5) reduces to (31) in the proof of Proposition 3 when $s = 2$. In what follows we show that $p_e/v_L^{k-1}(q^{k-1}) < \delta$ implies that there exist cutoffs $q^k$ and $\bar{q}^k$, payoff functions $v_B^k(q_0)$ and $v_L^k(q_0)$, and a map $Q^k_+(q_0)$ such that (A1) to (A8) are also satisfied when $s = k$.

From the discussion of 2–step equilibria in the main text, it is easy to see that the following conditions are necessary and sufficient for a $k$–step equilibrium to exist:

$$q^+ \left( q_0, v_L^{k-1}(q') \right) = q';$$
$$q' \in [q^{k-1}, \bar{q}^{k-1}] \cap (0, 1);$$
$$\pi^B_h(q_0) < \pi^B_\ell \left( q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q') \right).$$

(33)

The first of the above three conditions is condition (19) in the main text. We first show that (19) and (33) imply (34), so that the first two conditions are necessary and sufficient for a $k$–step equilibrium to exist. In order to do so, assume that (19) and (33) hold and let

$$\eta^k(q_0, \delta) = \pi^B_h(q_0) - \pi^B_\ell \left( q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q') \right).$$

Note that

$$\pi^B_\ell \left( q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q') \right) = (1 - q_0)F \left( \frac{p_e}{v_L^{k-1}(q')} \right) [u_L - p_e] + \delta \left\{ q_0 + (1 - q_0) \left[ 1 - F \left( \frac{p_e}{v_L^{k-1}(q')} \right) \right] \right\} \pi^B_h(q')$$
$$+ \delta \left\{ q_0 + (1 - q_0) \left[ 1 - F \left( \frac{p_e}{v_B^{k-1}(q')} \right) \right] \right\} \left[ v_B^{k-1}(q') - \pi^B_h(q') \right]$$
$$= \delta \pi^B_h(q_0) + (1 - q_0)F \left( \frac{p_e}{v_L^{k-1}(q')} \right) [u_L - p_e - \delta (u_L - p_h)]$$
$$+ \delta \left\{ q_0 + (1 - q_0) \left[ 1 - F \left( \frac{p_e}{v_L^{k-1}(q')} \right) \right] \right\} \left[ v_B^{k-1}(q') - \pi^B_h(q') \right],$$

where the last equality follows from (29) in the proof of Lemma 4. Similarly, one can show that if $q'' = Q^k_+(q')$, then

$$\pi^B_\ell \left( q', \delta, v_L^{k-2}(q''), v_B^{k-2}(q'') \right) = \delta \pi^B_h(q') + (1 - q')F \left( \frac{p_e}{v_L^{k-2}(q'')} \right) [u_L - p_e - \delta (u_L - p_h)]$$
$$+ \delta \left\{ q' + (1 - q') \left[ 1 - F \left( \frac{p_e}{v_L^{k-2}(q'')} \right) \right] \right\} \left[ v_B^{k-2}(q'') - \pi^B_h(q'') \right].$$
Now observe that

\[ v_B^{k-1}(q') - \pi_h^B(q') = \int \max \{ 0, \pi_h^B(q') - \pi_l^B(q', \delta, v_B^{k-2}(q''), v_B^{k-2}(q''')) \} \, dF(\delta) \]

\[ \geq \int \max \{ 0, \pi_h^B(q''') - \pi_l^B(q'', \delta, v_B^{k-3}(Q^k-2(q'''), v_B^{k-2}(Q^k+2(q'''))) \} \, dF(\delta) \]

\[ = v_B^{k-2}(q'') - \pi_h^B(q''), \]

where the inequality follows from (A5). Therefore,

\[ \pi_l^B (q_0, \delta, v_l^{k-1}(q'), v_B^{k-1}(q')) - \pi_l^B (q', \delta, v_B^{k-2}(q''), v_B^{k-2}(q''')) \]

\[ \geq \delta \left[ \pi_h^B(q_0) - \pi_h^B(q') \right] + \lambda \left[ (1 - q_0) F \left( \frac{p_\ell}{v_l^{k-1}(q')} \right) - (1 - q') F \left( \frac{p_\ell}{v_B^{k-2}(q'')} \right) \right], \]

where \( \lambda = \{ u_L - p_\ell - \delta(u_L - p_h) - \delta [v_B^{k-1}(q'') - \pi_h^B(q'')] \} \geq u_L - p_\ell - \delta v_B^{k-1}(q''') > 0 \). Given that \( v_B^{k-1}(q') < v_B^{k-2}(q'') \) by (A6) and that \( q' \geq q_0 \), we can then conclude that

\[ \pi_l^B (q', \delta, v_l^{k-2}(q''), v_B^{k-2}(q''')) - \pi_l^B (q_0, \delta, v_B^{k-1}(q'), v_B^{k-1}(q')) < \pi_h^B(q') - \pi_h^B(q_0). \]

Consequently, \( \eta^{k-1}(q', \delta) \geq \eta^k(q_0, \delta) \) for all \( \delta \in [\delta, \overline{\delta}] \). In particular, since \( \eta^{k-1}(q', \overline{\delta}) \leq 0 \) for all \( q' \in [q^{k-1}, \overline{q}^{k-1}] \) \( \cap \) \( (0, 1) \), we have that \( \eta^k(q_0, \overline{\delta}) \leq 0 \) as well, so that (34) is indeed satisfied.

Suppose now that \( p_\ell/v_L^{k-1}(q^{k-1}) < \overline{\delta} \) and define the cutoffs \( q^k \) and \( \overline{q}^k \) to be such that: (i) \( q^+(q^k, v_B^{k-1}(q^{k-1})) = q^{k-1} \) if \( p_\ell/v_L^{k-1}(q^{k-1}) < \overline{\delta} \) and \( q^k = 0 \) otherwise; (ii) \( q^+(\overline{q}^k, v_B^{k-1}(\overline{q}^{k-1})) = \overline{q}^{k-1} \). It is immediate to see \( 0 < q^k < \overline{q}^{k-1} \). Since \( q^{k-1} = 0 \) if, and only if, \( p_\ell/v_B^{k-2}(q^{k-2}) \geq \overline{\delta} \) (by (A8)) and \( v_B^{k-1}(q^{k-1}) \leq v_B^{k-2}(q^{k-2}) \) (by (A6)), we have that \( \overline{q}^k \leq q^{k-1} \). Now note that if \( q^{k-1} = 0 \), then (trivially) \( \overline{q}^k > q^{k-1} \). Suppose then that \( q^{k-1} > 0 \). Given that \( \overline{q}^{k-1} > q^{k-2} \), we have that \( v_B^{k-1}(\overline{q}^{k-1}) = v_B^{k-2}(\overline{q}^{k-2}) \geq v_B^{k-2}(q^{k-2}) \). Thus,

\[ q^+(\overline{q}^k, v_B^{k-2}(q^{k-2})) \geq q^+(\overline{q}^k, v_B^{k-1}(\overline{q}^{k-1})) = \overline{q}^{k-1} > q^{k-2} = q^+(q^{k-1}, v_B^{k-2}(q^{k-2})), \]

from which we obtain that \( \overline{q}^k > q^{k-1} \); recall that \( q^+(q, v_L) \) is strictly increasing in \( q \) when \( p_\ell/v_L < \overline{\delta} \). Finally, the same argument used in the proof of Proposition 3—just replace the superscripts “1” and “2” with “\( k - 1 \)” and “\( k \)” respectively—shows that: (i) there exists a \( k \)-step equilibrium if, and only if, \( q_0 \in [q^k, \overline{q}^k] \cap (0, 1) \); (ii) for each \( q_0 \in [q^k, \overline{q}^k] \cap (0, 1) \), there exists a unique \( q' = Q^k(q_0) \in [q^{k-1}, \overline{q}^{k-1}] \cap (0, 1) \) such that \( q' \) is the value of \( q_1 \) in any \( k \)-step
equilibrium when the initial fraction of type $H$ sellers is $q_0$; (iii) the map $Q^k_+$ is continuous and strictly increasing. Thus, (A1), (A2), (A3), (A5), and (A8) are valid for $s = k$.

To finish the induction step, let $v^k_B$ and $v^k_L$ be given by (20) and (21), respectively, where $\xi^k(q_0) = \int \{ \eta^k(q_0, \delta) \geq 0 \} dF(\delta)$. By construction, for each $q_0 \in [q^k, \bar{q}^k) \cap (0, 1)$, $v^k_B(q_0)$ and $v^k_L(q_0)$ are, respectively, the payoffs to buyers and type $L$ sellers in a $k$–step equilibrium (so that (A4) holds when $s = k$), and $\xi^k(q_0)$ is the fraction of buyers who offer $p_h$ in the first period of trade in a $k$–step equilibrium. The same argument used in the proof of Lemma 5 shows that $\xi^k(q_0)$ is increasing in $q_0$ and that (A6) and (A7) hold when $s = k$—once again just replace the superscripts “1” and “2” with “$k − 1$” and “$k$,” respectively.

The induction process described above continues until $k$ is such that $p_\ell/v^k_L(\bar{q}^k) \geq \delta$, if such a $k$ exists. We conclude the proof by showing that such a $k$ indeed exists, so that $K = \max\{ k : p_\ell/v^{k-1}_L(\bar{q}^{k-1}) < \delta \}$. Suppose not. In this case, there exists a strictly decreasing sequence $\{ q^k \}_{k=0}^{\infty}$ such that if $q_0 < q^k$, then there exists a $s$–step equilibrium with $s \geq k$ when the initial fraction of type $H$ sellers in the market is $q_0$. Since the market clears in a finite number of periods in any equilibrium, it must then be that $\lim_{k \to \infty} q^k = 0$. In particular, there exists $k_0 \in \mathbb{N}$ such that $\pi^B(\bar{q}^k) < 0$ for all $k \geq k_0$. This implies that $\xi^{k-1}(\bar{q}^k) = 0$ for all $k \geq k_0$, as not even a myopic buyer finds it optimal to offer $p_h$ when the expected payoff from doing so is negative. Therefore, $\lim_{k \to \infty} v^k_L(\bar{q}^k) = \lim_{k \to \infty} v^{k-1}_L(\bar{q}^k) = p_\ell$, a contradiction. ■

**Time to Market Clearing (Section 4)**

Let $q^*$ be such that $\pi^B(q^*) = 0$. Clearly $q^0 > q^*$ regardless of $F$, as the payoff from offering $p_\ell$ is positive, and so no buyer offers $p_h$ when $q \leq q^*$. Suppose then that $q_0 < q^*$ and let $N \geq 1$ be such that $\bar{\delta}^N p_h > p_\ell \geq \bar{\delta}^{N+1} p_h$. Now define $\{ v^k_L \}_{k=0}^N$ to be the sequence such that $v^0_L = p_h$ and $v^k_L = \int \max\{ p_\ell, \delta v^{k-1}_L \} dF(\delta)$ for all $k \in \{1, \ldots, N\}$. By construction, the payoff to sellers in a $k$–step equilibrium is bounded below by $v^k_L$. Since $v^k_L$ is decreasing in $k$, the fraction of high quality assets in the market after $N$ periods of trade is bounded above by

$$q_{\text{max},N} = \frac{q_0}{q_0 + (1 - q_0) [1 – F(p_\ell/v^{N-1}_L)]^N}.$$
Thus, the market takes at least \( N \) periods to clear if \( q_{\text{max}, N} \leq q^* \), which holds if

\[
1 - F \left( \frac{p_t}{L_{L-1}^N} \right)^N > \frac{q_0(1 - q^*)}{(1 - q_0)q^*}.
\]

(35)

Note that the right side of (35) is smaller than one since \( q_0 < q^* \). Finally, given that \( \frac{L_{L-1}^N}{\delta N-1} \geq \frac{p_h}{\delta N-1} \), we have that \( F(p_t/\delta N-1 p_h) \) converges to zero, in which case the left side of (35) converges to one. Therefore, if the distribution \( F \) puts sufficient mass on discount factors close enough to \( \delta \), then the market takes at least \( N \) periods to clear when \( q_0 < q^* \). It is easy to see that there are values of the model’s parameters for which \( N \) can be very large.

**Lemmas 6 and 7 and Proofs**

**Lemma 6.** \( E^k_I(q_0) \) is decreasing in \( q_0 \) for all \( k \in \{0, \ldots, K\} \).

**Proof:** For each \( q_0 \in [q^k, \bar{q}^k) \cap (0, 1) \), let \( \Lambda_{q_0}^k : \{0, \ldots, k\} \to [0, 1] \) be the c.d.f. given by

\[
\Lambda_{q_0}^k(s) = \sum_{r=0}^s \lambda^k(s|q_0).
\]

By construction, \( \Lambda_{q_0}^k(s) \) is the probability that a type \( H \) seller trades his asset on or before period \( s \in \{0, \ldots, k\} \) in a \( k \)-step equilibrium when the initial fraction of high quality assets is \( q_0 \). A straightforward induction argument shows that

\[
\Lambda_{q_0}^k(s) = 1 - \prod_{r=0}^s \left[ 1 - \xi^{k-r}(q_r) \right];
\]

recall that \( \{q_t\}_{t=1}^k \) is the sequence such that \( q_t = Q^{k-t+1}_+(q_{t-1}) \) for all \( t \in \{1, \ldots, k\} \). We know from the main text that an increase in \( q_0 \) increases \( \xi^{k-r}(q_r) \) for all \( r \in \{0, \ldots, k\} \). Thus, \( q_0 < q'_0 \) in \( [q^k, \bar{q}^k) \cap (0, 1) \) implies that \( \Lambda_{q_0}^k(s) \geq \Lambda_{q'_0}^k(s) \) for all \( s \in \{0, \ldots, k\} \), in which case \( \Lambda_{q_0}^k \) dominates \( \Lambda_{q'_0}^k \) in the first–order stochastic sense. The desired result follows immediately from this last fact.

**Lemma 7.** \( E_I(q_0) \) is decreasing in \( q_0 \).

**Proof:** Let \( q_0, q'_0 \in (0, 1) \) be such that \( q_0 < q'_0 \). By construction, there exist \( k_1, k_2 \in \{0, \ldots, K\} \) such that \( E_I(q_0) = E^k_{H^k}(q_0) \) and \( E_I(q'_0) = E^k_{H^k}(q'_0) \). There are two cases to consider.
(i) Suppose first that \( k_2 \geq k_1 \). In this case, there exists a \( k_1 \)-step equilibrium at \( q_0' \). Indeed, if \( q_0' \geq q^{k_1} \), then \( q_0' \geq q^{k_2} \), which implies that no \( k_2 \)-step equilibrium exists at \( q_0' \), a contradiction. Moreover, if \( q_0' < q^{k_1} \), then \( q_0 < q^{k_1} \), in which case no \( k_1 \)-step equilibrium exists at \( q_0 \), a contradiction as well. By Lemma 6, we then have that \( E(q_0') \leq E^{k_1}_H(q_0') \leq E^{k_1}_H(q_0) = E(q_0) \);

(ii) Suppose now that \( k_2 < k_1 \) and let \( k' \) be the greatest value of \( k \) such that a \( k \)-step equilibrium exists at \( q_0' \). Note that \( k' \geq k_2 \) and \( q^{k'+1} \leq q_0' \). Moreover, note that if \( k' \geq k_1 \), then \( E(q_0') \leq E^{k_1}_H(q_0) \leq E^{k_1}_H(q_0) = E(q_0) \). Suppose then that \( k' < k_1 \). We know from the proof of Theorem 1 that \( \lim_{q_0 \to q^k} \xi^k(q_0) = \xi^{k-1}(q^k) \) for all \( k \in \{1, \ldots, K\} \), from which it is easy to see that \( \lim_{q_0 \to q^k} E^{k}_H(q_0) = E^{k-1}_H(q^k) \) for all \( k \in \{1, \ldots, K\} \). Hence, by repeatedly using Lemma 6, we have that

\[
E^{k_1-1}_H(q^{k_1}) \geq \lim_{q_0 \to q^{k_1-1}} E^{k_1-1}_H(q_0) = E^{k_1-2}_H(q^{k_1-1}) \geq \cdots \geq E^{k'}_H(q^{k'+1}).
\]

Thus, using Lemma 6 one more time, we can conclude that

\[
E(q_0) \geq \lim_{q_0 \to q^{k_1}} E^{k}_H(q_0) = E^{k_1-1}_H(q^{k_1}) \geq E^{k'}_H(q^{k'+1}) \geq E^{k'}_H(q_0') \geq E(q_0').
\]

This establishes the desired result.

\( \blacksquare \)
Supplemental Appendix (not for publication)

Here we establish that all of the basic properties of equilibria established in Section 3 remain true in the more general environment where buyers are free to offer any price to sellers. In addition to relaxing the restriction that buyers can offer one of two prices, in this section we also place no restrictions on either $\delta$ and $\bar{\delta}$.

Strategies and Equilibria

A pricing rule for a buyer is a function $p^B : [\delta, \bar{\delta}] \rightarrow [0, u_H]$ such that $p^B(\delta)$ is the price the buyer offers if his discount factor is $\delta$; no buyer has an incentive to offer a price greater than $u_H$. An acceptance rule for a seller is a function $a : [\delta, \bar{\delta}] \times [0, u_H] \rightarrow \{0, 1\}$ such that $a(\delta, p) = 1$ if, and only if, the seller accepts an offer of $p$ when his discount factor is $\delta$. A pure strategy for a buyer is a sequence $\{p^B_t\}_{t=0}^{\infty}$, where $p^B_t$ is his pricing rule in period $t$, while a pure strategy for a type $j \in \{L, H\}$ seller is a sequence $\{a^j_t\}_{t=0}^{\infty}$, where $a^j_t$ is his acceptance rule in period $t$. The same argument as in the main text shows that neither buyers nor sellers have any reason to condition their behavior on their past histories.

We again consider symmetric pure-strategy equilibria. As in the main text, for any strategy profile $\sigma = (\{p^B_t\}, \{a^L_t\}, \{a^H_t\})$, we let $A^j_t(p|\sigma)$ be the probability that a type $j$ seller in the market in period $t$ accepts an offer $p$, $T(\sigma)$ be the period in which the market clears, and $q_t(\sigma)$ be the fraction of type $H$ sellers in the market in period $t \in \{0, \ldots, T(\sigma)\}$. Note that $A^j_t(p|\sigma) = \int a^j_t(\delta, p)dF(\delta)$ and that $\{q_t(\sigma)\}_{t=0}^{T(\sigma)}$ satisfies the law of motion (8). The definition of an equilibrium is the same as before, except that now prices are fully flexible.\footnote{We compute payoffs for the buyers and both types of sellers using the same refinement as before.}

Basic Results

Here, we establish a number of results that will be useful when we prove that the market clears in finite time in any equilibrium. The first result we prove is that a seller accepts any
price offer greater than \( c_H \). Since this implies that it is never optimal for a buyer to offer more than \( c_H \), the payoff to type \( H \) sellers is zero in any equilibrium.

**Lemma 8.** A seller accepts any price offer \( p > c_H \).

**Proof:** First note that no buyer offers more than \( u_H \). Hence, sellers accept any price offer \( p > p^* = \max\{c_H, \delta u_H\} \). This, however, implies that no buyer offers a price greater than \( p^* \).

Now define \( \{p_n\}_{n=1}^{\infty} \) to be such that \( p_1 = p^* \) and \( p_{n+1} = \max\{c_H, \delta p_n\} \) for all \( n \geq 1 \). It is easy to see that if no buyer offers more than \( p_n \), then sellers accept any price offer \( p > p_{n+1} \), which, in turn, implies that no buyer offers more than \( p_{n+1} \). The desired result now follows from the fact that \( \{p_n\} \) converges to \( c_H \). \( \square \)

Since a seller always accepts an offer if indifferent between accepting or rejecting it, Lemma 8 pins down the behavior of the type \( H \) sellers: they accept an offer \( p \) if, and only if, \( p \geq c_H \). From now on we take the behavior of the type \( H \) sellers as given and focus on the behavior of buyers and type \( L \) sellers. An immediate corollary of the proof of Lemma 8 is that a type \( L \) seller accepts any offer greater than \( \delta c_H \) regardless of his discount factor. Hence, it is never optimal for a buyer to make an offer \( p \in (p_\ell, c_H) \), where \( p_\ell = \min\{\delta c_H, u_L\} \).

Denote by \( \pi^B(p, \delta, q, v_L, v_B) \) the payoff to a buyer who offers \( p \) when: (i) his discount factor is \( \delta \); (ii) the fraction of high quality assets in the market is \( q \); (iii) the continuation payoff to a type \( L \) seller who chooses not to trade is \( v_L \in [0, c_H] \); and (iv) the continuation payoff to the buyer should he not trade is \( v_B \geq 0 \). Given that sellers always accept an offer of \( c_H \), we have that

\[
\pi^B(c_H, \delta, q, v_L, v_B) = q[u_H - c_H] + (1 - q)[u_L - c_H].
\]

Now observe that since only type \( L \) sellers can possibly accept an offer smaller than \( c_H \), we have that if \( p \in [0, c_H) \), then

\[
\pi^B(p, \delta, q, v_L, v_B) = (1 - q)A^L(p)[u_L - p] + \left\{q + (1 - q) \left[1 - A^L(p)\right]\right\} \delta v_B = (1 - q)A^L(p)[u_L - p - \delta v_B] + \delta v_B,
\]
where we slightly abuse notation and let

$$A_L(p) = \int \mathbb{I}\{\delta : p \geq \delta v_L\} dF(\delta)$$

denote the probability that a type \(L\) seller accepts \(p < c_H\). Note that \(A_L\) is continuous and strictly increasing in \(p\) for all \(p \in [\delta v_L, \delta v_L]\), so long as \(v_L > 0\).

Fix \(q \in (0,1)\), \(v_L \in [0,c_H]\), and \(v_B \geq 0\), and let \(\Psi : [\delta, \delta] \rightarrow [0,p_B] \cup \{c_H\}\) be such that

$$\Psi(\delta) = \text{argmax}_{p \in [0,p_B] \cup \{c_H\}} \pi_B(p, \delta, q, v_L, v_B).$$

A selection from \(\Psi\) is a map \(p_B : [\delta, \delta] \rightarrow [0,p_L] \cup \{c_H\}\) such that \(p_B(\delta) \in \Psi(\delta)\) for all \(\delta \in [\delta, \delta]\).

Lemma 9 below says that if both \(v_L\) and \(v_B\) are positive, then any selection from \(\Psi\) is non decreasing in \(\delta\) as long as it is optimal for the buyer to trade with positive probability.

**Lemma 9.** Fix \(q \in (0,1)\) and suppose that \(v_L, v_B > 0\). There exists \(\delta \leq \delta^* \leq \delta\) such that \(A_L(p) > 0\) for all \(p \in \Psi(\delta)\) when \(\delta < \delta^*\) and \(A_L(p) = 0\) for all \(p \in \Psi(\delta)\) when \(\delta > \delta^*\). Moreover, any selection from \(\Psi\) is non increasing in \(\delta\) when \(\delta \in (\delta, \delta^*)\).

**Proof:** We divide the proof in two cases.

(1) Consider first the case in which \(\pi_B(c_H, \delta, q, v_L, v_B) \leq 0\), so that it is not optimal for a buyer with \(\delta > \delta^*\) to offer \(p = c_H\); since \(v_B > 0\), the buyer’s payoff from offering \(p \leq \delta v_L\) and trading with zero probability is \(\delta v_B\), which is positive by assumption. We are done if \(u_L \leq \delta(v_L + v_B)\), so that

$$(1 - q)A_L(p)[u_L - p - \delta v_B] + \delta v_B < \delta v_B$$

for all \(p > \delta v_L\), in which case any buyer with \(\delta > \delta^*\) strictly prefers not to trade. Suppose then that \(u_L > \delta(v_L + v_B)\), and let

$$\delta^* = \sup\{\delta \in (\delta, \delta] : u_L > \delta v_L + \delta v_B\}.$$ 

By construction, if \(\delta > \delta^*\), then it is optimal for the buyer to offer \(p \leq \delta v_L\) and not trade. Now observe that if \(\delta < \delta^*\), then it is optimal for the buyer to offer \(p \in (\delta v_L, u_L - \delta v_B)\) and trade with positive probability. This establishes the first part of the lemma.
In order to establish the second part of the lemma, let \( p^B \) be a selection from \( \Psi \) and assume that \( u_L > \delta(v_L + v_B) \). First, note that if \( p \in \Psi(\delta) \) is such that \( A^L(p) > 0 \) and \( p < c_H \), then \( p < \delta v_L \); otherwise, the buyer would be able to increase his payoff by lowering \( p \) without affecting the probability of trade. Now let \( \delta_1 < \delta_2 < \delta^* \) and note that

\[
A^L(p^B(\delta_1))[u_L - p^B(\delta_1) - \delta_1 v_B]
\]

\[
= A^L(p^B(\delta_1))[u_L - p^B(\delta_1) - \delta_2 v_B] + A^L(p^B(\delta_1))(\delta_2 - \delta_1) v_B
\]

\[
\leq A^L(p^B(\delta_2))[u_L - p^B(\delta_2) - \delta_2 v_B] + A^L(p^B(\delta_1))(\delta_2 - \delta_1) v_B
\]

\[
= A^L(p^B(\delta_2))[u_L - p^B(\delta_2) - \delta_1 v_B] + [A^L(p^B(\delta_1)) - A^L(p^B(\delta_2))](\delta_2 - \delta_1) v_B.
\]

Given that \( A^L(p^B(\delta_1))[u_L - p^B(\delta_1) - \delta_1 v_B] \geq A^L(p^B(\delta_2))[u_L - p^B(\delta_2) - \delta_1 v_B] \) and \( v_B > 0 \), we then have that \( A^L(p^B(\delta_1)) \geq A^L(p^B(\delta_2)) \). Since \( A^L(p) \) is strictly increasing in \( p \) when \( p \in (\delta v_L, \delta v_L) \) and \( p^B(\delta_1), p^B(\delta_2) \in (\delta v_L, \delta v_L) \), we can then conclude that \( p^B(\delta_1) \geq p^B(\delta_2) \).

(2) Suppose now that \( \pi^B(c_H, \delta, q, v_L, v_B) > 0 \). Let \( \Psi_\ell : [\delta, \delta^*] \rightarrow [0, \bar{p}_\ell] \) be such that

\[
\Psi_\ell(\delta) = \arg\max_{p \in [0, \bar{p}_\ell]} A^L(p)[u_L - p - \delta v_B]
\]

and define \( \pi^B_\ell(\delta, q, v_L, v_B) \) to be such that

\[
\pi^B_\ell(\delta, q, v_L, v_B) = \pi^B(\Psi_\ell(\delta), \delta, q, v_L, v_B).
\]

By construction, \( \pi^B_\ell(\delta, q, v_L, v_B) \) is the buyer’s payoff when he optimally offers \( p \in [0, \bar{p}_\ell] \). Now let \( \pi^B_h(q) = \pi^B(c_H, \delta, q, v_L, v_B) \). It is immediate to see that

\[
\Psi(\delta) = \begin{cases} 
\Psi_\ell(\delta) & \text{if } \pi^B_\ell(\delta, q, v_L, v_B) > \pi^B_h(q) \\
\Psi_\ell(\delta) \cup \{c_H\} & \text{if } \pi^B_\ell(\delta, q, v_L, v_B) = \pi^B_h(q)
\end{cases}
\]

\[
\text{if } \pi^B_\ell(\delta, q, v_L, v_B) < \pi^B_h(q)
\]

The desired results follow from Case 1 if we show that the indirect utility function \( \pi^B_\ell \) is strictly increasing in \( \delta \). This, however, follows immediately from the fact that \( \pi^B \) is strictly increasing in \( \delta \) if \( v_B > 0 \).

Notice that if \( F \) is concave, then \( \Psi(\delta) \) is single-valued for all \( \delta \in [\delta, \delta^*] \). A straightforward consequence of the above result is that even when \( F \) is not concave, \( \Psi(\delta) \) is single-valued for almost all \( \delta \in [\delta, \delta^*] \).
**Market Clearing**

We know from the discussion following Lemma 8 that the market clears in the first period in which all buyers in the market offer $c_H$. We also know that if $\sigma$ is an equilibrium, then $a_t^H(\delta, p) \leq a_t^L(\delta, p)$ for all $t \geq 0$, $\delta \in [\delta, \bar{\delta})$, and $p \in [0, u_H]$; as in the case with two fixed prices, type $H$ sellers are effectively more patient than type $L$ sellers. Hence,

$$
\int A_t^L(p^B_t(\delta)|\sigma)dF(\delta) = \int \int a_t^L(\delta, p_t^B(\delta_b))dF(\delta_b)dF(\delta_s) \\
\geq \int \int a_t^H(\delta, p_t^B(\delta_b))dF(\delta_b)dF(\delta_s) = \int A_t^H(p^B_t(\delta)|\sigma)dF(\delta)
$$

for all $t \geq 0$, which implies that Lemma 2 in Section 3 also holds with fully flexible prices:

**Lemma 10.** If $\sigma$ is an equilibrium, then $\{q_t(\sigma)\}_{t=0}^{T(\sigma)}$ is increasing.

The next result shows that the market clears immediately when the initial fraction of high quality assets is sufficiently close to one. For this, let $q^{**}$ be the value of $q_0$ such that

$$
q_0u_H + (1 - q_0)u_L - c_H = \max\{u_L, \bar{\delta}[u_H - c_H]\}.
$$

Note that $q^{**} < 1$ since $u_H - c_H > u_L$.

**Lemma 11.** The market clears immediately when $q_0 > q^{**}$.

**Proof:** Suppose that $q_0 > q^{**}$ and consider a buyer. The buyer’s payoff if he offers $c_H$ is

$$
q_0u_H + (1 - q_0)u_L - c_H > \pi^* = \max\{u_L, \bar{\delta}[u_H - c_H]\}.
$$

Now observe that since $u_H - c_H$ is the greatest payoff possible for the buyer, an upper bound on his payoff if he offers $p < c_H$ is

$$
\bar{\pi} = (1 - q_0)A^L(p)[u_L - p] + \{q_0 + (1 - q_0)(1 - A^L(p))\}\bar{\delta}[u_H - c_H].
$$

Given that $\bar{\pi} \leq \pi^*$, we can then conclude that it is not optimal for the buyer to offer less than $c_H$. Thus, the market clears immediately when $q_0 > q^{**}$.  

We can now establish that the market clears in finite time in any equilibrium.
Proposition 4. The market clears in finite time in any equilibrium.

Proof: Suppose not and consider an equilibrium in which the market never clears. Let: (i) \( q_t \) be the fraction of type \( H \) sellers in the market in period \( t \); (ii) \( v_t^L \) be the lifetime payoff to a type \( L \) seller in the market in period \( t \); (iii) \( v_t^B \) be the lifetime payoff to a buyer in the market in period \( t \); and (iv) \( p_t^B \) be the pricing rule of the buyers in period \( t \). We know that \( \{q_t\} \) is convergent and, by Lemma 11, that its limit \( q_\infty \) is smaller than one. We also know that \( \forall t \geq 0 \) \( v_t^L > 0 \). Indeed, \( v_t^L = 0 \) only if all buyers in the market offer \( p = 0 \) in period \( t \) and in all subsequent periods. This, however, implies that any buyer with a discount factor greater than zero can profitably deviate in period \( t \) by offering a small positive price. Let \( q^* \in (0, 1) \) be such that \( \pi_h^B(q^*) = 0 \). There are two cases to consider: \( q_\infty > q^* \) and \( q_\infty \leq q^* \).

(1) Suppose that \( q_\infty \leq q^* \). Since \( q_t < q^* \) for all \( t \geq 0 \), it is never optimal for a buyer to offer \( c_H \). Hence, \( p_t^B(\delta) \leq \bar{p}_t \) for all \( \delta \in [\delta, \bar{\delta}] \) and all \( t \geq 0 \). In particular, the mass of type \( H \) sellers who trade in each period is zero. Now let \( A_t^L(p) = F(p/v_t^{L+1}) \) denote the probability that a type \( L \) seller accepts an offer \( p \in [0, \bar{p}_t] \) in period \( t \); we know from above that \( v_t^L > 0 \) for all \( t \geq 0 \). Moreover, let \( \chi_t = \int A_t^L(p_t^B(\delta))dF(\delta) \) be the ex–ante probability of trade in period \( t \); by assumption, trade only takes place between buyers and type \( L \) sellers. The law of motion (8) becomes

\[
q_{t+1} = \frac{q_t}{q_t + (1 - q_t)(1 - \chi_t)}.
\]

First, note that a necessary condition for \( q_\infty < 1 \) is that \( \{\chi_t\} \) converges to zero. Now, observe that \( \lim_{t \to \infty} \chi_t = 0 \) implies that both \( \{v_t^L\} \) and \( \{v_t^B\} \) converge to zero. This makes intuitive sense, as \( \chi_t \) is the ex–ante probability of trade in period \( t \). To finish, note that since \( \lim_{t \to \infty} v_t^L = 0 \), there exists \( t_0 \in \mathbb{Z}_+ \) such that if \( t \geq t_0 \), then the probability that an offer of \( u_L/2 \) is accepted by a type \( L \) seller is at least 1/2. Thus, \( t \geq t_0 \) implies that \( v_t^B \geq (1 - q_t)u_L/4 \), in which case \( \lim_{t \to \infty} v_t^B \geq (1 - q_\infty)u_L/4 > 0 \), a contradiction.

(2) Suppose now that \( q_\infty > q^* \). We can assume, without loss of generality, that \( q_0 > q^* \). Let \( \Psi_{t,t}: [\delta, \bar{\delta}] \Rightarrow [0, \bar{p}_t] \) be such that

\[
\Psi_{t,t}(\delta) = \arg\max_{p \in [0, \bar{p}_t]} A_t^L(p)[u_L - p - \delta v_t^{B+1}]
\]
and define $\pi^B_{\ell,t}(\delta)$ to be such that

$$\pi^B_{\ell,t}(\delta) = \pi^B(\Psi_{\ell,t}(\delta), \delta, q_t, v^L_{t+1}, v^B_{t+1}).$$

Given that $v^B_t > 0$ for all $t \geq 0$, as a buyer can always offer $p = c_H$ and obtain a payoff greater than $\pi^B_h(q_0)$, the same argument as in the proof of Lemma 9 shows that the indirect utility function $\pi^B_{\ell,t}$ is strictly increasing in $\delta$.

Now, let $\pi^B_{h,t} = \pi^B_h(q_t)$ and define $\delta^*_t$ to be such that

$$\delta^*_t = \inf\{\delta \in [\delta, \delta] : \pi^B_{\ell,t}(\delta) \geq \pi^B_{h,t}\}.$$

Since $\pi^B_{\ell,t}(\delta) > \pi^B_{h,t}$ for all $t \geq 0$, otherwise the market clears in finite time, we have that $\delta^*_t \in [\delta, \delta]$ for all $t \geq 0$. By construction, offering $p = c_H$ in period $t$ is strictly optimal if, and only if, $\delta < \delta^*_t$. Let then

$$\tilde{\gamma}_t = \int_{\delta^*_t}^{\bar{\delta}} A^L_t(p^B_t(\delta))dF(\delta)$$

and $\gamma_t = [1 - F(\delta^*_t)]^{-1}\tilde{\gamma}_t$. The law of motion (8) becomes

$$q_{t+1} = \frac{q_t[1 - F(\delta^*_t)]}{q_t[1 - F(\delta^*_t)] + (1 - q_t)[1 - F(\delta^*_t) - \tilde{\gamma}_t]}$$

$$= \frac{q_t}{q_t + (1 - q_t)[1 - \gamma_t]},$$

(36)

Since, by the Maximum Theorem, $\pi^B_{\ell,t}(\delta)$ is continuous in $\delta$, it is easy to show that $\{\delta^*_t\}$ is a convergent sequence. Denote its limit by $\delta^*_\infty$. There are two possibilities: $\delta^*_\infty < \bar{\delta}$ or $\delta^*_\infty = \bar{\delta}$.

(2.1) Suppose first that $\delta^*_\infty < \bar{\delta}$. A necessary condition for $q_\infty < 1$ is that $\{\gamma_t\}$ converges to zero. This, however, is only possible if $\{\tilde{\gamma}_t\}$ converges to zero, which implies that $\{v^B_t\}$ converges to $v^B_\infty < \pi^B_h(q_\infty)$. This makes intuitive sense, as $\tilde{\gamma}_t$ is the ex–ante probability that a buyer transacts in period $t$ conditional on the event that he makes an offer $p < c_H$ and $\delta^*_\infty < \bar{\delta}$ implies that in every period $t$ the probability that a buyer in the market offers $p < c_H$ is bounded away from zero. On the other hand, an option for a buyer in the market in period $t$ is to offer $p = c_H$ regardless of his discount factor. This implies that $v^B_t \geq \pi^B_h(q_t)$ for all $t \geq 0$, which, in turn, implies that $v^B_\infty \geq \pi^B_h(q_\infty)$, a contradiction.

(2.2) Suppose then that $\delta^*_\infty = \bar{\delta}$. In this case, we have that $v^B_\infty = \pi^B_h(q_\infty)$, as the ex–ante probability that a buyer in the market offers $p < c_H$ converges to zero in the long–run. Now,
let $\varepsilon > 0$ be such that $\varepsilon u_L < (1 - \bar{\delta})\pi^B_h(q_\infty)/4$. Moreover, let $t_0 \in \mathbb{Z}_+$ be such that

$$v_t^B \leq (1 + \bar{\delta})\pi^B_h(q_\infty)/2\bar{\delta} \quad \text{and} \quad (3 + \bar{\delta})\pi^B_h(q_\infty)/4 < \pi^B_h(q_t).$$

for all $t \geq t_0$. By construction, $t \geq t_0$ and $A_t^L(p) \leq \varepsilon$ imply that

$$(1 - q_t)A_t^L(p)[u_L - p] + \{q_t + (1 - q_t)[1 - A_t^L(p)]\}\delta v_{t+1}^B < \pi^B_h(q_t).$$

Thus, when $t \geq t_0$, any price offer $p \in [0, \overline{p}_t]$ such that $A_t^L(p) \leq \varepsilon$ is not optimal for a buyer regardless of his discount factor.

Assume that $t \geq t_0$, so that $A_t^L(p_t^B(\delta)) > 0$ for all $\delta \in [\delta^*_t, \bar{\delta}]$ by the previous paragraph. Lemma 9 then implies that that $p_t^B$ is non increasing in $\delta$. So, $\lim_{\delta \to \pi_t^B} p_t^B(\delta)$ exists for all $t \geq 0$. Denote this limit by $\alpha_t$. Since $F$ has no mass points and is strictly increasing in its support, we then have that

$$\tilde{\chi}_t = \int_{\delta_t^*}^{\bar{\delta}} A_t^L(p_t^B(\delta))dF(\delta) \geq A_t^L(\alpha_t)[1 - F(\delta^*_t)],$$

where once again we make use of Lemma 9. Therefore, the law of motion (36) implies that

$$q_{t+1} \geq \frac{q_t}{q_t + (1 - q_t)[1 - A_t^L(\alpha_t)]},$$

from which we obtain that $\{A_t^L(\alpha_t)\}$ converges to zero (otherwise $q_\infty = 1$). This, however, implies that there exists $t_1 \geq t_0$ such that $A_t^L(\alpha_t) \leq \varepsilon/2$ when $t \geq t_1$. Consequently, given that $F$ is continuous, for all $t \geq t_1$, there exists $\delta' \in [\delta^*_t, \bar{\delta})$ such that if $\delta \in (\delta', \bar{\delta})$, then $A_t^L(p_t^B(\delta)) \leq \varepsilon$. In other words, when $t$ is large, a positive mass of buyers behaves sub–optimally, a contradiction. This completes the proof. ■
References


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Figure 4: The Dynamics of Trade