Suboptimal risk placement based on inverse volatility weighting

Using volatility as the risk measure for portfolio's risk, a very simple asset allocation approach is presented so that risk is equally split into all portfolio's components without optimization

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Introduction

Asset allocation is key for successful Portfolio Management. Authors such as Grignold and Kahn (2000); Campbell and Viceira (2002), among others, state that asset allocation is the single most important component for performance.

Classical mean-variance was introduced by Markowitz's work (1959) and established the need for 2 inputs: risk and return. When the objective (utility) function is quadratic, it's unconstrained solution is well known and given by an allocation vector $\vec{w} = \frac{1}{\gamma}\Sigma^{-1}E[\vec{R}]$ that depends on the risk aversion coefficient γ, the variance-covariance matrix Σ and vector of expected returns $E[\vec{R}]$.

An alternative approach is to focus on risk-only allocation, that is independent of expected return. One example is the so-called risk-parity (Roncalli,2012) where the marginal contribution of each asset to total risk is the same. In this short study, we're proposing a simpler solution that relies on the concept of risk-parity but without optimization.

Suboptimal risk parity allocation

Let's assume that an investor has a portfolio of "n" different instruments. Portfolio's variance is given by:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j \qquad (1)$$

Assuming that this investor believes in market efficiency, he/she decides to control portfolio's risk only, measured by volatility, provided the market price of risk[2] is given.

In this context, the investor decides to establish the following rules to build the portfolio:

(i) Risk is measured by volatility;

(ii) As initial step, for all instruments "i", the product $w_i\sigma_i$ has the same value;

(iii) $\sum_{i=1}^{n} w_i = L; L > 0$

(iv) Correlations need to satisfy:
$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j > 0$ or
$n + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{ij} > 0$

The initial allocation, as given by Rule (ii), implies that each instrument´s direct contribution to risk is the same; when there is no leverage. Rule (iii) means that weights sum up 1.

Using (ii), $w_i\sigma_i = w_j\sigma_j; i \neq j$, and expanding (1):

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j = \sigma_i^2 w_i^2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\right)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij} = n + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{ij}$$

$$\sigma_p^2 = \sigma_i^2 w_i^2\left(n + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{ij}\right); i = 1,\dots,n \quad (2)$$

Also, from rule (ii):

$$w_j\sigma_j = w_i\sigma_i \Rightarrow w_j = \frac{w_1\sigma_1}{\sigma_j} \quad (3)$$

Assuming leverage L=k and (3), it implies that:

$$\sum_{i=1}^{n} w_i = k \Rightarrow \sum_{j=1}^{n}\frac{w_1\sigma_1}{\sigma_j} = k = w_1\sigma_1\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_n}\right) \quad (4)$$

Provided $w_1\sigma_1 = w_i\sigma_i$, then, from (4), instruments' weights are given by:

$$w_i = \frac{k \times \frac{1}{\sigma_i}}{\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_n}\right)} = \frac{k \times \frac{1}{\sigma_i}}{\sum_{j=1}^{n}\frac{1}{\sigma_j}} \quad (5)$$

Based on result (5), allocation should be proportional to the inverse of volatility. If assets are uncorrelated, portfolio's risk, given equations (2) and (5), is:

$$\sigma_p = k\frac{\sqrt{\left(n + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{ij}\right)}}{\sum_{j=1}^{n}\frac{1}{\sigma_j}} = k\frac{\sqrt{n}}{\sum_{j=1}^{n}\frac{1}{\sigma_j}} \quad (6)$$

It should be observed that, in the specific case of a given leverage "k", risk is a consequence of the choices made at allocation level as implied by (6).

Another way to see this result, (6), is to look at it from the opposite perspective: given a desired portfolio risk, what should be asset allocation? The answer, also from (6), is obvious: it corresponds to fine-tuning k so that the target volatility level is achieved.

Rule (iv) is needed to have positive variance; its "second" form is less generic and applies when Rule (ii) was already obtained.

Application

Let's assume one has a portfolio of 3 assets:

Table 1: parameters of portfolio's assets

	Correlation matrix			Volatility
	A1	A2	A3	
A1	1.00	0.00	0.50	40%
A2	0.00	1.00	0.50	30%
A3	0.50	0.50	1.00	50%

In this case, from equation (6);

$$\sqrt{\left(n + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{ij}\right)} = \sqrt{3 + 2(0 + 0.5 + 0.5))}$$
$$= 2.24$$

$$\sum_{j=1}^{n}\frac{1}{\sigma_j}=\frac{1}{0.4}+\frac{1}{0.3}+\frac{1}{0.5}=7.83$$

$$\sigma_p=k\frac{\sqrt{\left(n+2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{ij}\right)}}{\sum_{j=1}^{n}\frac{1}{\sigma_j}}=k\frac{2.24}{7.83}\approx 0.3k$$

If there is no leverage, then $\sigma_p \approx 30\%$ and the weights are given by (5):

$$w_i=\frac{k\times\frac{1}{\sigma_i}}{\sum_{j=1}^{n}\frac{1}{\sigma_j}}\Rightarrow w_1=\frac{2.50}{7.83};w_2=\frac{3.33}{7.83};w_3=\frac{2.00}{7.83}$$

If an investor wants a portfolio that has volatility of 3%, then the weights should be divided by 10 and the remaining cash should be invested into a risk-free asset.

Portfolio's volatility can be written (Roncalli, 2012) as:

$$\sigma(\vec{w})=\sqrt{\vec{w}^T\Sigma\vec{w}} \tag{7}$$

To obtain individual contributions:

$$\overrightarrow{RC}=\frac{\partial\sigma(\vec{w})}{\partial\vec{w}}=\frac{\Sigma\vec{w}}{\sqrt{\vec{w}^T\Sigma\vec{w}}} \tag{8}$$

Thus (7) can be rewritten as:

$$\sigma(\vec{w})=\frac{\Sigma\vec{w}}{\sqrt{\vec{w}^T\Sigma\vec{w}}}\cdot\vec{w}=\sum_{i=1}^{n}\left(\frac{\Sigma\vec{w}}{\sqrt{\vec{w}^T\Sigma\vec{w}}}\right)_i w_i \tag{9}$$

This is known as the Euler decomposition formula, where the product is between row "i" of both vectors. Considering table 1 as well as the weights when k=1, we have the following covariance matrix:

$$\Sigma=$$

$$=\begin{bmatrix}40\% & 0 & 0\\ 0 & 30\% & 0\\ 0 & 0 & 50\%\end{bmatrix}^T\begin{bmatrix}1 & 0 & 0.5\\ 0 & 1 & 0.5\\ 0.5 & 0.5 & 1\end{bmatrix}\begin{bmatrix}40\% & 0 & 0\\ 0 & 30\% & 0\\ 0 & 0 & 50\%\end{bmatrix}$$

$$=\begin{bmatrix}0.160 & 0 & 0.100\\ 0 & 0.090 & 0.075\\ 0.100 & 0.075 & 0.250\end{bmatrix}$$

Therefore, using this covariance matrix and the vector of weights, $\vec{w}=\begin{bmatrix}\frac{2.50}{7.83}\\ \frac{3.33}{7.83}\\ \frac{2.00}{7.83}\end{bmatrix}=\begin{bmatrix}0.319\\ 0.425\\ 0.255\end{bmatrix}$, risk contributions per unit of asset are:

$$\begin{bmatrix}RC_1\\ RC_2\\ RC_3\end{bmatrix}=\frac{\Sigma\vec{w}}{\sqrt{\vec{w}^T\Sigma\vec{w}}}=\frac{\begin{bmatrix}0.0766\\ 0.0574\\ 0.1277\end{bmatrix}}{28{,}55\%}=\begin{bmatrix}0.268\\ 0.201\\ 0.447\end{bmatrix}$$

Multiplying the risk contributions, "RCs", by the weights, the following risk participations, "RPs", are obtained:

$$\begin{bmatrix}RP_{A1}\\ RP_{A2}\\ RP_{A3}\end{bmatrix}=\begin{bmatrix}0.268\\ 0.201\\ 0.447\end{bmatrix}\cdot\begin{bmatrix}0.319\\ 0.425\\ 0.255\end{bmatrix}=\begin{bmatrix}0.09\\ 0.09\\ 0.11\end{bmatrix}\therefore\begin{bmatrix}30\%\\ 30\%\\ 40\%\end{bmatrix}$$

As expected, the risk contribution of each asset is not the same (ERC or the *equal* contribution to risk approach) because correlations are not zero. However, after this initial step, to obtain the intended contribution of $\frac{1}{n}$ or 33.3%, the weights should be adjusted accordingly - if the result is not in the acceptable range:

$$\frac{adjust}{RC}=\frac{(desired-actual)}{RC_1}=\frac{0.10-0.09}{0.268}=+5\%$$

$$\frac{(desired-actual)}{RC_2}=\frac{0.10-0.09}{0.201}=+7\%$$

$\frac{(desired-actual)}{RC_3} = \frac{0.10-0.11}{0.447} = -3\%$

Comparing the total required increases in weights to the total reductions, we see that the lowest absolute weight adjust is 3% Thus this must be the basis for the adjustment. Taking from instrument 3 and splitting between instruments 1 and 2, proportional to ratio $\frac{adjust}{RC}$, respectively:

$$w_1 = w_{1,in} + adjust_1 = \frac{2.50}{7.83} + \frac{5\%}{5\%+7\%} \times 3\% = 33.3\%;$$

$$w_2 = w_{2,in} + adjust_2 = \frac{3.33}{7.83} + \frac{7\%}{5\%+7\%} \times 3\% = 44.4\%;$$

$$w_3 = w_{3,in} + adjust_3 = \frac{2.00}{7.83} - 3\% = 22.4\%$$

Recalculating risk contributions using adjusted weights:

$$\begin{bmatrix} RC_1 \\ RC_2 \\ RC_3 \end{bmatrix} = \frac{\Sigma\vec{w}}{\sqrt{\vec{w}^T\Sigma\vec{w}}} = \frac{\begin{bmatrix} 0.0756 \\ 0.0567 \\ 0.123 \end{bmatrix}}{27{,}87\%} = \begin{bmatrix} 0.271 \\ 0.203 \\ 0.439 \end{bmatrix}$$

Finally, risk participations are close to desired (1/3 each):

$$\begin{bmatrix} RP_{A1} \\ RP_{A2} \\ RP_{A3} \end{bmatrix} = \begin{bmatrix} 0.271 \\ 0.203 \\ 0.439 \end{bmatrix} \cdot \begin{bmatrix} 0.333 \\ 0.444 \\ 0.224 \end{bmatrix} = \begin{bmatrix} 0.09 \\ 0.09 \\ 0.10 \end{bmatrix} \therefore \begin{bmatrix} 32\% \\ 32\% \\ 35\% \end{bmatrix}$$

Concluding remarks

In the context of asset allocation via Risk Budgeting, as in Roncalli (2012) for example, we concluded that it can be implemented without optimization.

Even though in this short study, as in other works, the focus was on risk-parity - meaning a risk allocation *equally* split into all risky components - the same approach can be applied to establish other required risk allocations, by simply changing equation (3) and targeting to define required relative contributions. Finally, the results presented in this short study can be applied at risk factor level by simply substituting assets for factors and adjusting covariance accordingly.

Bibliography

MARKOWITZ,H.: Portfolio Selection:Efficient diversification of investments; *Wiley*;1959.

GRIGNOLD, R.C; KAHN,R.N.: Active Portfolio Management: a quantitative approach for producing superior returns and controlling risk;2ed; *Irwin*;NY;NY;2000.

CAMPBELL,J.Y;VICEIRA,L.: Startegic Asset Allocation: Portfolio choice for long-term investors;*Oxford*;NY;NY;2002.

RONCALLI, T.: From Portfolio Optimization to Risk Parity; in 6th R / Rmetrics Meielisalp Workshop & Summer School on Computational Finance and Financial Engineering; Meielisalp; Switzerland;2012.

Notes

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[2]Market price of risk is assumed to be the ratio between portfolio's excess return and its correspondent risk: $\lambda = \frac{r_p - r_f}{\sigma}$